CHAPTER 8
Laplace Transforms

IN THIS CHAPTER we study the method of *Laplace transforms*, which illustrates one of the basic problem solving techniques in mathematics: transform a difficult problem into an easier one, solve the latter, and then use its solution to obtain a solution of the original problem. The method discussed here transforms an initial value problem for a constant coefficient equation into an algebraic equation whose solution can then be used to solve the initial value problem. In some cases this method is merely an alternative procedure for solving problems that can be solved equally well by methods that we considered previously; however, in other cases the method of Laplace transforms is more efficient than the methods previously discussed. This is especially true in physical problems dealing with discontinuous forcing functions.

SECTION 8.1 defines the Laplace transform and developes its properties.

SECTION 8.2 deals with the problem of finding a function that has a given Laplace transform.

SECTION 8.3 applies the Laplace transform to solve initial value problems for constant coefficient second order differential equations on $(0, \infty)$.

SECTION 8.4 introduces the unit step function.

SECTION 8.5 uses the unit step function to solve constant coefficient equations with piecewise continuous forcing functions.

SECTION 8.6 deals with the convolution theorem, an important theoretical property of the Laplace transform.

SECTION 8.7 introduces the idea of impulsive force, and treats constant coefficient equations with impulsive forcing functions.

SECTION 8.8 is a brief table of Laplace transforms.

8.1 INTRODUCTION TO THE LAPLACE TRANSFORM

Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If g is integrable over the interval $[a, T]$ for every $T > a$, then the *improper integral of g over* $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\,dt = \lim_{T\to\infty} \int_a^T g(t)\,dt. \tag{8.1.1}$$

We say that the improper integral *converges* if the limit in (8.1.1) exists; otherwise, we say that the improper integral *diverges* or *does not exist*. Here's the definition of the Laplace transform of a function f.

Definition 8.1.1 Let f be defined for $t \ge 0$ and let s be a real number. Then the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{8.1.2}$$

for those values of s for which the improper integral converges.

It is important to keep in mind that the variable of integration in (8.1.2) is t, while s is a parameter independent of t. We use t as the independent variable for f because in applications the Laplace transform is usually applied to functions of time.

The Laplace transform can be viewed as an operator L that transforms the function $f = f(t)$ into the function $F = F(s)$. Thus, (8.1.2) can be expressed as

$$F = L(f).$$

The functions f and F form a *transform pair*, which we'll sometimes denote by

$$f(t) \leftrightarrow F(s).$$

It can be shown that if $F(s)$ is defined for $s = s_0$ then it's defined for all $s > s_0$ (Exercise 14**(b)**).

Computation of Some Simple Laplace Transforms

Example 8.1.1 Find the Laplace transform of $f(t) = 1$.

Solution From (8.1.2) with $f(t) = 1$,

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \int_0^T e^{-st}\,dt.$$

If $s \ne 0$ then

$$\int_0^T e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^T = \frac{1 - e^{-sT}}{s}. \tag{8.1.3}$$

Therefore

$$\lim_{T\to\infty} \int_0^T e^{-st}\,dt = \begin{cases} \dfrac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases} \tag{8.1.4}$$

If $s = 0$ the integrand reduces to the constant 1, and

$$\lim_{T\to\infty} \int_0^T 1\,dt = \lim_{T\to\infty} \int_0^T 1\,dt = \lim_{T\to\infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad s > 0.$$

This result can be written in operator notation as

$$L(1) = \frac{1}{s}, \quad s > 0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s > 0.$$

REMARK: It is convenient to combine the steps of integrating from 0 to T and letting $T \to \infty$. Therefore, instead of writing (8.1.3) and (8.1.4) as separate steps we write

$$\int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \begin{cases} \dfrac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases}$$

We'll follow this practice throughout this chapter.

Example 8.1.2 Find the Laplace transform of $f(t) = t$.

Solution From (8.1.2) with $f(t) = t$,

$$F(s) = \int_0^\infty e^{-st} t\,dt. \tag{8.1.5}$$

If $s \ne 0$, integrating by parts yields

$$\begin{aligned} \int_0^\infty e^{-st} t\,dt &= -\frac{te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = -\left[\frac{t}{s} + \frac{1}{s^2}\right]e^{-st}\Big|_0^\infty \\ &= \begin{cases} \dfrac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases} \end{aligned}$$

If $s = 0$, the integral in (8.1.5) becomes

$$\int_0^\infty t\,dt = \frac{t^2}{2}\Big|_0^\infty = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$L(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

Example 8.1.3 Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution From (8.1.2) with $f(t) = e^{at}$,

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt.$$

Combining the exponentials yields

$$F(s) = \int_0^\infty e^{-(s-a)t}\, dt.$$

However, we know from Example 8.1.1 that

$$\int_0^\infty e^{-st}\, dt = \frac{1}{s}, \quad s > 0.$$

Replacing s by $s - a$ here shows that

$$F(s) = \frac{1}{s-a}, \quad s > a.$$

This can also be written as

$$L(e^{at}) = \frac{1}{s-a}, \quad s > a, \quad \text{or} \quad e^{at} \leftrightarrow \frac{1}{s-a}, \quad s > a.$$

Example 8.1.4 Find the Laplace transforms of $f(t) = \sin \omega t$ and $g(t) = \cos \omega t$, where ω is a constant.

Solution Define

$$F(s) = \int_0^\infty e^{-st} \sin \omega t\, dt \tag{8.1.6}$$

and

$$G(s) = \int_0^\infty e^{-st} \cos \omega t\, dt. \tag{8.1.7}$$

If $s > 0$, integrating (8.1.6) by parts yields

$$F(s) = -\frac{e^{-st}}{s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t\, dt,$$

so

$$F(s) = \frac{\omega}{s} G(s). \tag{8.1.8}$$

If $s > 0$, integrating (8.1.7) by parts yields

$$G(s) = -\frac{e^{-st} \cos \omega t}{s} \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t\, dt,$$

so

$$G(s) = \frac{1}{s} - \frac{\omega}{s} F(s).$$

Now substitute from (8.1.8) into this to obtain

$$G(s) = \frac{1}{s} - \frac{\omega^2}{s^2} G(s).$$

Solving this for $G(s)$ yields

$$G(s) = \frac{s}{s^2 + \omega^2}, \quad s > 0.$$

This and (8.1.8) imply that

$$F(s) = \frac{\omega}{s^2 + \omega^2}, \quad s > 0.$$

Tables of Laplace transforms

Extensive tables of Laplace transforms have been compiled and are commonly used in applications. The brief table of Laplace transforms in the Appendix will be adequate for our purposes.

Example 8.1.5 Use the table of Laplace transforms to find $L(t^3e^{4t})$.

Solution The table includes the transform pair

$$t^n e^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}}.$$

Setting $n = 3$ and $a = 4$ here yields

$$L(t^3e^{4t}) = \frac{3!}{(s-4)^4} = \frac{6}{(s-4)^4}.$$

We'll sometimes write Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms.

Linearity of the Laplace Transform

The next theorem presents an important property of the Laplace transform.

Theorem 8.1.2 *[Linearity Property] Suppose $L(f_i)$ is defined for $s > s_i$, $1 \le i \le n$). Let s_0 be the largest of the numbers $s_1, s_2, \dots, s_n$, and let $c_1, c_2, \dots, c_n$ be constants. Then*

$$L(c_1f_1 + c_2f_2 + \cdots + c_nf_n) = c_1L(f_1) + c_2L(f_2) + \cdots + c_nL(f_n) \text{ for } s > s_0.$$

Proof We give the proof for the case where $n = 2$. If $s > s_0$ then

$$\begin{aligned} L(c_1f_1 + c_2f_2) &= \int_0^\infty e^{-st}\left(c_1f_1(t) + c_2f_2(t)\right)\, dt \\ &= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt \\ &= c_1L(f_1) + c_2L(f_2). \end{aligned}$$

Example 8.1.6 Use Theorem 8.1.2 and the known Laplace transform

$$L(e^{at}) = \frac{1}{s-a}$$

to find $L(\cosh bt)$ $(b \ne 0)$.

Solution By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned} L(\cosh bt) &= L\left(\frac{1}{2}e^{bt} + \frac{1}{2}e^{-bt}\right) \\ &= \frac{1}{2}L(e^{bt}) + \frac{1}{2}L(e^{-bt}) \qquad \text{(linearity property)} \\ &= \frac{1}{2}\frac{1}{s-b} + \frac{1}{2}\frac{1}{s+b}, \end{aligned} \tag{8.1.9}$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in (8.1.9) yields

$$L(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

The First Shifting Theorem

The next theorem enables us to start with known transform pairs and derive others. (For other results of this kind, see Exercises 6 and 13.)

Theorem 8.1.3 [*First Shifting Theorem*] *If*

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{8.1.10}$$

is the Laplace transform of $f(t)$ for $s > s_0$, then $F(s-a)$ is the Laplace transform of $e^{at} f(t)$ for $s > s_0 + a$.

PROOF. Replacing s by $s - a$ in (8.1.10) yields

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt \tag{8.1.11}$$

if $s - a > s_0$; that is, if $s > s_0 + a$. However, (8.1.11) can be rewritten as

$$F(s-a) = \int_0^\infty e^{-st}\left(e^{at} f(t)\right)\, dt,$$

which implies the conclusion.

Example 8.1.7 Use Theorem 8.1.3 and the known Laplace transforms of 1, t, $\cos\omega t$, and $\sin\omega t$ to find

$$L(e^{at}), \quad L(te^{at}), \quad L(e^{\lambda t}\sin\omega t), \text{ and } L(e^{\lambda t}\cos\omega t).$$

Solution In the following table the known transform pairs are listed on the left and the required transform pairs listed on the right are obtained by applying Theorem 8.1.3.

$$f(t) \leftrightarrow F(s) \qquad\qquad e^{at}f(t) \leftrightarrow F(s-a)$$

$$
\begin{array}{rlrl}
1 \leftrightarrow \dfrac{1}{s}, & s > 0 & e^{at} \leftrightarrow \dfrac{1}{(s-a)}, & s > a \\
t \leftrightarrow \dfrac{1}{s^2}, & s > 0 & te^{at} \leftrightarrow \dfrac{1}{(s-a)^2}, & s > a \\
\sin \omega t \leftrightarrow \dfrac{\omega}{s^2+\omega^2}, & s > 0 & e^{\lambda t}\sin \omega t \leftrightarrow \dfrac{\omega}{(s-\lambda)^2+\omega^2}, & s > \lambda \\
\cos \omega t \leftrightarrow \dfrac{s}{s^2+\omega^2}, & s > 0 & e^{\lambda t}\sin \omega t \leftrightarrow \dfrac{s-\lambda}{(s-\lambda)^2+\omega^2}, & s > \lambda
\end{array}
$$

Existence of Laplace Transforms

Not every function has a Laplace transform. For example, it can be shown (Exercise 3) that

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \infty$$

for every real number s. Hence, the function $f(t) = e^{t^2}$ does not have a Laplace transform.

Our next objective is to establish conditions that ensure the existence of the Laplace transform of a function. We first review some relevant definitions from calculus.

Recall that a limit

$$\lim_{t\to t_0} f(t)$$

exists if and only if the one-sided limits

$$\lim_{t\to t_0-} f(t) \quad \text{and} \quad \lim_{t\to t_0+} f(t)$$

both exist and are equal; in this case,

$$\lim_{t\to t_0} f(t) = \lim_{t\to t_0-} f(t) = \lim_{t\to t_0+} f(t).$$

Recall also that f is continuous at a point t_0 in an open interval (a,b) if and only if

$$\lim_{t\to t_0} f(t) = f(t_0),$$

which is equivalent to

$$\lim_{t\to t_0+} f(t) = \lim_{t\to t_0-} f(t) = f(t_0). \tag{8.1.12}$$

For simplicity, we define

$$f(t_0+) = \lim_{t\to t_0+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t\to t_0-} f(t),$$

so (8.1.12) can be expressed as

$$f(t_0+) = f(t_0-) = f(t_0).$$

If $f(t_0+)$ and $f(t_0-)$ have finite but distinct values, we say that f has a *jump discontinuity* at t_0, and

$$f(t_0+) - f(t_0-)$$

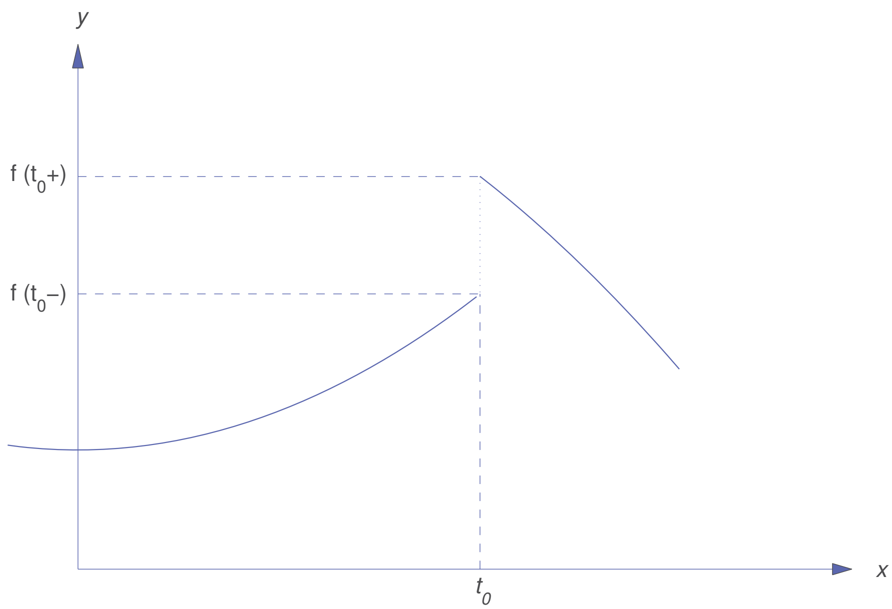

Figure 8.1.1 A jump discontinuity

is called the *jump* in f at t_0 (Figure 8.1.1).

If $f(t_0+)$ and $f(t_0-)$ are finite and equal, but either f isn't defined at t_0 or it's defined but

$$f(t_0) \neq f(t_0+) = f(t_0-),$$

we say that f has a *removable discontinuity* at t_0 (Figure 8.1.2). This terminolgy is appropriate since a function f with a removable discontinuity at t_0 can be made continuous at t_0 by defining (or redefining)

$$f(t_0) = f(t_0+) = f(t_0-).$$

REMARK: We know from calculus that a definite integral isn't affected by changing the values of its integrand at isolated points. Therefore, redefining a function f to make it continuous at removable discontinuities does not change $L(f)$.

Definition 8.1.4

(i) A function f is said to be *piecewise continuous* on a finite closed interval $[0, T]$ if $f(0+)$ and $f(T-)$ are finite and f is continuous on the open interval $(0, T)$ except possibly at finitely many points, where f may have jump discontinuities or removable discontinuities.

(ii) A function f is said to be *piecewise continuous* on the infinite interval $[0, \infty)$ if it's piecewise continuous on $[0, T]$ for every $T > 0$.

Figure 8.1.3 shows the graph of a typical piecewise continuous function.

It is shown in calculus that if a function is piecewise continuous on a finite closed interval then it's integrable on that interval. But if f is piecewise continuous on $[0, \infty)$, then so is $e^{-st}f(t)$, and therefore

$$\int_0^T e^{-st} f(t)\, dt$$

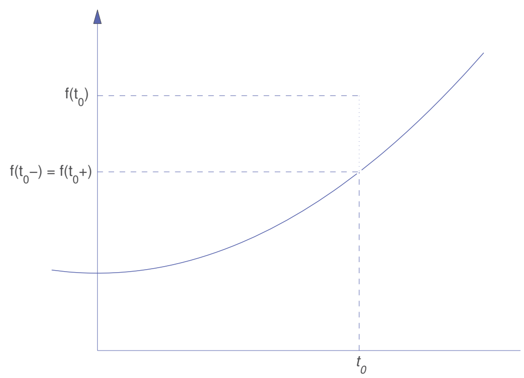

Figure 8.1.2

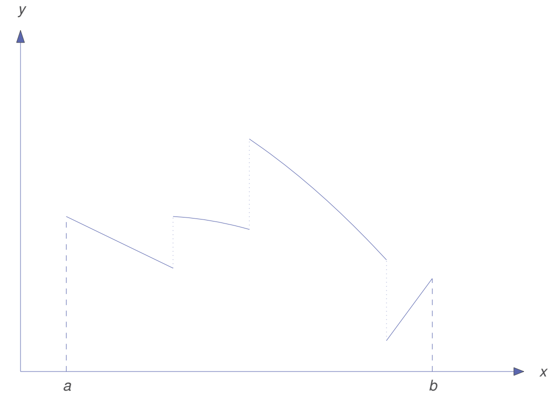

Figure 8.1.3 A piecewise continuous function on $[a, b]$

exists for every $T > 0$. However, piecewise continuity alone does not guarantee that the improper integral

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt \tag{8.1.13}$$

converges for s in some interval (s_0, ∞). For example, we noted earlier that (8.1.13) diverges for all s if $f(t) = e^{t^2}$. Stated informally, this occurs because e^{t^2} increases too rapidly as $t \to \infty$. The next definition provides a constraint on the growth of a function that guarantees convergence of its Laplace transform for s in some interval (s_0, ∞) .

Definition 8.1.5 A function f is said to be *of exponential order* s_0 if there are constants M and t_0 such that

$$|f(t)| \le Me^{s_0 t}, \quad t \ge t_0. \tag{8.1.14}$$

In situations where the specific value of s_0 is irrelevant we say simply that f is *of exponential order*.

The next theorem gives useful sufficient conditions for a function f to have a Laplace transform. The proof is sketched in Exercise 10.

Theorem 8.1.6 *If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then $L(f)$ is defined for $s > s_0$.*

REMARK: We emphasize that the conditions of Theorem 8.1.6 are sufficient, but *not necessary*, for f to have a Laplace transform. For example, Exercise 14**(c)** shows that f may have a Laplace transform even though f isn't of exponential order.

Example 8.1.8 If f is bounded on some interval $[t_0, \infty)$, say

$$|f(t)| \le M, \quad t \ge t_0,$$

then (8.1.14) holds with $s_0 = 0$, so f is of exponential order zero. Thus, for example, $\sin \omega t$ and $\cos \omega t$ are of exponential order zero, and Theorem 8.1.6 implies that $L(\sin \omega t)$ and $L(\cos \omega t)$ exist for $s > 0$. This is consistent with the conclusion of Example 8.1.4.

Example 8.1.9 It can be shown that if $\lim_{t\to\infty} e^{-s_0t} f(t)$ exists and is finite then f is of exponential order s_0 (Exercise 9). If α is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order s_0, since

$$\lim_{t\to\infty} e^{-s_0t} t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \ge 0$, f is also continuous on $[0, \infty)$. Therefore Exercise 9 and Theorem 8.1.6 imply that $L(t^\alpha)$ exists for $s \ge s_0$. However, since s_0 is an arbitrary positive number, this really implies that $L(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 8.1.2 and Exercises 6 and 8.

Example 8.1.10 Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -3e^{-t}, & t \ge 1. \end{cases}$$

Solution Since f is defined by different formulas on $[0, 1)$ and $[1, \infty)$, we write

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}(1)\, dt + \int_1^\infty e^{-st}(-3e^{-t})\, dt.$$

Since

$$\int_0^1 e^{-st}\, dt = \begin{cases} \dfrac{1-e^{-s}}{s}, & s \ne 0, \\ 1, & s = 0, \end{cases}$$

and

$$\int_1^\infty e^{-st}(-3e^{-t})\, dt = -3\int_1^\infty e^{-(s+1)t}\, dt = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \dfrac{1-e^{-s}}{s} - 3\dfrac{e^{-(s+1)}}{s+1}, & s > -1,\ s \ne 0, \\ 1 - \dfrac{3}{e}, & s = 0. \end{cases}$$

This is consistent with Theorem 8.1.6, since

$$|f(t)| \le 3e^{-t}, \quad t \ge 1,$$

and therefore f is of exponential order $s_0 = -1$.
REMARK: In Section 8.4 we'll develop a more efficient method for finding Laplace transforms of piecewise continuous functions.

Example 8.1.11 We stated earlier that

$$\int_0^\infty e^{-st} e^{t^2}\, dt = \infty$$

for all s, so Theorem 8.1.6 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{s_0t}} = \lim_{t\to\infty} \frac{1}{M} e^{t^2 - s_0t} = \infty,$$

so

$$e^{t^2} > Me^{s_0t}$$

for sufficiently large values of t, for any choice of M and s_0 (Exercise 3).

8.1 Exercises

1. Find the Laplace transforms of the following functions by evaluating the integral $F(s) = \int_0^\infty e^{-st} f(t)\,dt$.

(a) t **(b)** te^{-t} **(c)** $\sinh bt$
(d) $e^{2t} - 3e^t$ **(e)** t^2

2. Use the table of Laplace transforms to find the Laplace transforms of the following functions.

(a) $\cosh t \sin t$ **(b)** $\sin^2 t$ **(c)** $\cos^2 2t$
(d) $\cosh^2 t$ **(e)** $t \sinh 2t$ **(f)** $\sin t \cos t$
(g) $\sin\left(t + \frac{\pi}{4}\right)$ **(h)** $\cos 2t - \cos 3t$ **(i)** $\sin 2t + \cos 4t$

3. Show that

$$\int_0^\infty e^{-st} e^{t^2}\,dt = \infty$$

for every real number s.

4. Graph the following piecewise continuous functions and evaluate $f(t+)$, $f(t-)$, and $f(t)$ at each point of discontinuity.

(a) $f(t) = \begin{cases} -t, & 0 \le t < 2, \\ t-4, & 2 \le t < 3, \\ 1, & t \ge 3. \end{cases}$ **(b)** $f(t) = \begin{cases} t^2+2, & 0 \le t < 1, \\ 4, & t = 1, \\ t, & t > 1. \end{cases}$

(c) $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2, \\ 2\sin t, & \pi/2 \le t < \pi, \\ \cos t, & t \ge \pi. \end{cases}$ **(d)** $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2, & t = 1, \\ 2-t, & 1 \le t < 2, \\ 3, & t = 2, \\ 6, & t > 2. \end{cases}$

5. Find the Laplace transform:

(a) $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1, \\ e^{-2t}, & t \ge 1. \end{cases}$ **(b)** $f(t) = \begin{cases} 1, & 0 \le t < 4, \\ t, & t \ge 4. \end{cases}$

(c) $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$ **(d)** $f(t) = \begin{cases} te^t, & 0 \le t < 1, \\ e^t, & t \ge 1. \end{cases}$

6. Prove that if $f(t) \leftrightarrow F(s)$ then $t^k f(t) \leftrightarrow (-1)^k F^{(k)}(s)$. HINT: *Assume that it's permissible to differentiate the integral $\int_0^\infty e^{-st} f(t)\,dt$ with respect to s under the integral sign.*

7. Use the known Laplace transforms

$$L(e^{\lambda t}\sin \omega t) = \frac{\omega}{(s-\lambda)^2 + \omega^2} \quad \text{and} \quad L(e^{\lambda t}\cos \omega t) = \frac{s-\lambda}{(s-\lambda)^2 + \omega^2}$$

and the result of Exercise 6 to find $L(te^{\lambda t}\cos \omega t)$ and $L(te^{\lambda t}\sin \omega t)$.

8. Use the known Laplace transform $L(1) = 1/s$ and the result of Exercise 6 to show that

$$L(t^n) = \frac{n!}{s^{n+1}}, \quad n = \text{integer}.$$

9. **(a)** Show that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0.
(b) Show that if f is of exponential order s_0 then $\lim_{t\to\infty} e^{-st} f(t) = 0$ for all $s > s_0$.

(c) Show that if f is of exponential order s_0 and $g(t) = f(t+\tau)$ where $\tau > 0$, then g is also of exponential order s_0.

10. Recall the next theorem from calculus.

THEOREM A. *Let g be integrable on $[0, T]$ for every $T > 0$. Suppose there's a function w defined on some interval $[\tau, \infty)$ (with $\tau \geq 0$) such that $|g(t)| \leq w(t)$ for $t \geq \tau$ and $\int_\tau^\infty w(t)\,dt$ converges. Then $\int_0^\infty g(t)\,dt$ converges.*

Use Theorem A to show that if f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then f has a Laplace transform $F(s)$ defined for $s > s_0$.

11. Prove: If f is piecewise continuous and of exponential order then $\lim_{s\to\infty} F(s) = 0$.

12. Prove: If f is continuous on $[0, \infty)$ and of exponential order $s_0 > 0$, then

$$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}L(f), \quad s > s_0.$$

HINT: *Use integration by parts to evaluate the transform on the left.*

13. Suppose f is piecewise continuous and of exponential order, and that $\lim_{t\to 0+} f(t)/t$ exists. Show that

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(r)\,dr.$$

HINT: *Use the results of Exercises* 6 *and* 11.

14. Suppose f is piecewise continuous on $[0, \infty)$.

(a) Prove: If the integral $g(t) = \int_0^t e^{-s_0\tau} f(\tau)\,d\tau$ satisfies the inequality $|g(t)| \leq M$ $(t \geq 0)$, then f has a Laplace transform $F(s)$ defined for $s > s_0$. HINT: *Use integration by parts to show that*

$$\int_0^T e^{-st} f(t)\,dt = e^{-(s-s_0)T} g(T) + (s - s_0)\int_0^T e^{-(s-s_0)t} g(t)\,dt.$$

(b) Show that if $L(f)$ exists for $s = s_0$ then it exists for $s > s_0$. Show that the function

$$f(t) = te^{t^2}\cos(e^{t^2})$$

has a Laplace transform defined for $s > 0$, even though f isn't of exponential order.

(c) Show that the function

$$f(t) = te^{t^2}\cos(e^{t^2})$$

has a Laplace transform defined for $s > 0$, even though f isn't of exponential order.

15. Use the table of Laplace transforms and the result of Exercise 13 to find the Laplace transforms of the following functions.

(a) $\dfrac{\sin \omega t}{t}$ $(\omega > 0)$ **(b)** $\dfrac{\cos \omega t - 1}{t}$ $(\omega > 0)$ **(c)** $\dfrac{e^{at} - e^{bt}}{t}$

(d) $\dfrac{\cosh t - 1}{t}$ **(e)** $\dfrac{\sinh^2 t}{t}$

16. The *gamma function* is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx,$$

which can be shown to converge if $\alpha > 0$.

(a) Use integration by parts to show that

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \quad \alpha>0.$$

(b) Show that $\Gamma(n+1)=n!$ if $n=1,2,3,\ldots$.

(c) From **(b)** and the table of Laplace transforms,

$$L(t^\alpha)=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad s>0,$$

if α is a nonnegative integer. Show that this formula is valid for any $\alpha>-1$. HINT: *Change the variable of integration in the integral for* $\Gamma(\alpha+1)$.

17. Suppose f is continuous on $[0,T]$ and $f(t+T)=f(t)$ for all $t\ge 0$. (We say in this case that f is *periodic with period* T.)

(a) Conclude from Theorem 8.1.6 that the Laplace transform of f is defined for $s>0$. HINT: *Since f is continuous on* $[0,T]$ *and periodic with period T, it's bounded on* $[0,\infty)$.

(b) **(b)** Show that

$$F(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt, \quad s>0.$$

HINT: *Write*

$$F(s)=\sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-st}f(t)\,dt.$$

Then show that

$$\int_{nT}^{(n+1)T} e^{-st}f(t)\,dt=e^{-nsT}\int_0^T e^{-st}f(t)\,dt,$$

and recall the formula for the sum of a geometric series.

18. Use the formula given in Exercise 17**(b)** to find the Laplace transforms of the given periodic functions:

(a) $f(t)=\begin{cases} t, & 0\le t<1,\\ 2-t, & 1\le t<2,\end{cases} \qquad f(t+2)=f(t), \quad t\ge 0$

(b) $f(t)=\begin{cases} 1, & 0\le t<\frac12,\\ -1, & \frac12\le t<1,\end{cases} \qquad f(t+1)=f(t), \quad t\ge 0$

(c) $f(t)=|\sin t|$

(d) $f(t)=\begin{cases} \sin t, & 0\le t<\pi,\\ 0, & \pi\le t<2\pi,\end{cases} \qquad f(t+2\pi)=f(t)$

8.2 THE INVERSE LAPLACE TRANSFORM

Definition of the Inverse Laplace Transform

In Section 8.1 we defined the Laplace transform of f by

$$F(s)=L(f)=\int_0^\infty e^{-st}f(t)\,dt.$$

We'll also say that f is an *inverse Laplace Transform* of F, and write

$$f=L^{-1}(F).$$

To solve differential equations with the Laplace transform, we must be able to obtain f from its transform F. There's a formula for doing this, but we can't use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we'll need.

Example 8.2.1 Use the table of Laplace transforms to find

$$\textbf{(a)}\ L^{-1}\left(\frac{1}{s^2-1}\right) \quad \text{and} \quad \textbf{(b)}\ L^{-1}\left(\frac{s}{s^2+9}\right).$$

SOLUTION(**a**) Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2-b^2}$$

shows that

$$L^{-1}\left(\frac{1}{s^2-1}\right) = \sinh t.$$

SOLUTION(**b**) Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2}$$

shows that

$$L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t. \blacksquare$$

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

Theorem 8.2.1 *[Linearity Property] If $F_1, F_2, \dots, F_n$ are Laplace transforms and $c_1, c_2, \dots, c_n$ are constants, then*

$$L^{-1}(c_1F_1 + c_2F_2 + \cdots + c_nF_n) = c_1L^{-1}(F_1) + c_2L^{-1}(F_2) + \cdots + c_nL^{-1}F_n.$$

Example 8.2.2 Find

$$L^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right).$$

Solution From the table of Laplace transforms in Section 8.8,,

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad \sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}.$$

Theorem 8.2.1 with $a = -5$ and $\omega = \sqrt{3}$ yields

$$\begin{aligned} L^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right) &= 8L^{-1}\left(\frac{1}{s+5}\right) + 7L^{-1}\left(\frac{1}{s^2+3}\right) \\ &= 8L^{-1}\left(\frac{1}{s+5}\right) + \frac{7}{\sqrt{3}}L^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right) \\ &= 8e^{-5t} + \frac{7}{\sqrt{3}}\sin\sqrt{3}t. \end{aligned}$$

Example 8.2.3 Find

$$L^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

Solution Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5}=\frac{3s+8}{(s+1)^2+4}.$$

Because of the form of the denominator, we consider the transform pairs

$$e^{-t}\cos 2t \leftrightarrow \frac{s+1}{(s+1)^2+4} \quad\text{and}\quad e^{-t}\sin 2t \leftrightarrow \frac{2}{(s+1)^2+4},$$

and write

$$\begin{aligned}L^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right) &= L^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right)+L^{-1}\left(\frac{5}{(s+1)^2+4}\right)\\ &= 3L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)+\frac{5}{2}L^{-1}\left(\frac{2}{(s+1)^2+4}\right)\\ &= e^{-t}(3\cos 2t+\frac{5}{2}\sin 2t).\end{aligned}$$

REMARK: We'll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 8.8.

Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s)=\frac{P(s)}{Q(s)},$$

where P and Q are polynomials in s with no common factors. Since it can be shown that $\lim_{s\to\infty}F(s)=0$ if F is a Laplace transform, we need only consider the case where degree$(P)<$ degree(Q). To obtain $L^{-1}(F)$, we find the partial fraction expansion of F, obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

Example 8.2.4 Find the inverse Laplace transform of

$$F(s)=\frac{3s+2}{s^2-3s+2}. \tag{8.2.1}$$

Solution (METHOD 1) Factoring the denominator in (8.2.1) yields

$$F(s)=\frac{3s+2}{(s-1)(s-2)}. \tag{8.2.2}$$

The form for the partial fraction expansion is

$$\frac{3s+2}{(s-1)(s-2)}=\frac{A}{s-1}+\frac{B}{s-2}. \tag{8.2.3}$$

Multiplying this by $(s-1)(s-2)$ yields

$$3s+2=(s-2)A+(s-1)B.$$

Setting $s=2$ yields $B=8$ and setting $s=1$ yields $A=-5$. Therefore

$$F(s)=-\frac{5}{s-1}+\frac{8}{s-2}$$

and

$$L^{-1}(F)=-5L^{-1}\left(\frac{1}{s-1}\right)+8L^{-1}\left(\frac{1}{s-2}\right)=-5e^{t}+8e^{2t}.$$

Solution (METHOD 2) We don't really have to multiply (8.2.3) by $(s-1)(s-2)$ to compute A and B. We can obtain A by simply ignoring the factor $s-1$ in the denominator of (8.2.2) and setting $s=1$ elsewhere; thus,

$$A=\left.\frac{3s+2}{s-2}\right|_{s=1}=\frac{3\cdot 1+2}{1-2}=-5. \tag{8.2.4}$$

Similarly, we can obtain B by ignoring the factor $s-2$ in the denominator of (8.2.2) and setting $s=2$ elsewhere; thus,

$$B=\left.\frac{3s+2}{s-1}\right|_{s=2}=\frac{3\cdot 2+2}{2-1}=8. \tag{8.2.5}$$

To justify this, we observe that multiplying (8.2.3) by $s-1$ yields

$$\frac{3s+2}{s-2}=A+(s-1)\frac{B}{s-2},$$

and setting $s=1$ leads to (8.2.4). Similarly, multiplying (8.2.3) by $s-2$ yields

$$\frac{3s+2}{s-1}=(s-2)\frac{A}{s-2}+B$$

and setting $s=2$ leads to (8.2.5). (It isn't necesary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in (8.2.4) and (8.2.5).)

The shortcut employed in the second solution of Example 8.2.4 is *Heaviside's method*. The next theorem states this method formally. For a proof and an extension of this theorem, see Exercise 10.

Theorem 8.2.2 *Suppose*

$$F(s)=\frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}, \tag{8.2.6}$$

where $s_1, s_2, \dots, s_n$ are distinct and P is a polynomial of degree less than n. Then

$$F(s)=\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}+\cdots+\frac{A_n}{s-s_n},$$

where A_i can be computed from (8.2.6) by ignoring the factor $s-s_i$ and setting $s=s_i$ elsewhere.

Example 8.2.5 Find the inverse Laplace transform of

$$F(s)=\frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)(s+1)}. \tag{8.2.7}$$

Solution The partial fraction expansion of (8.2.7) is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}. \tag{8.2.8}$$

To find A, we ignore the factor s in the denominator of (8.2.7) and set $s = 0$ elsewhere. This yields

$$A = \frac{6 + (1)(11)}{(-1)(-2)(1)} = \frac{17}{2}.$$

Similarly, the other coefficients are given by

$$B = \frac{6 + (2)(7)}{(1)(-1)(2)} = -10,$$

$$C = \frac{6 + 3(5)}{2(1)(3)} = \frac{7}{2},$$

and

$$D = \frac{6}{(-1)(-2)(-3)} = -1.$$

Therefore

$$F(s) = \frac{17}{2}\frac{1}{s} - \frac{10}{s-1} + \frac{7}{2}\frac{1}{s-2} - \frac{1}{s+1}$$

and

$$\begin{aligned} L^{-1}(F) &= \frac{17}{2}L^{-1}\left(\frac{1}{s}\right) - 10L^{-1}\left(\frac{1}{s-1}\right) + \frac{7}{2}L^{-1}\left(\frac{1}{s-2}\right) - L^{-1}\left(\frac{1}{s+1}\right) \\ &= \frac{17}{2} - 10e^{t} + \frac{7}{2}e^{2t} - e^{-t}. \end{aligned}$$

REMARK: We didn't "multiply out" the numerator in (8.2.7) before computing the coefficients in (8.2.8), since it wouldn't simplify the computations.

Example 8.2.6 Find the inverse Laplace transform of

$$F(s) = \frac{8 - (s+2)(4s+10)}{(s+1)(s+2)^2}. \tag{8.2.9}$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}. \tag{8.2.10}$$

Because of the repeated factor $(s+2)^2$ in (8.2.9), Heaviside's method doesn't work. Instead, we find a common denominator in (8.2.10). This yields

$$F(s) = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}. \tag{8.2.11}$$

If (8.2.9) and (8.2.11) are to be equivalent, then

$$A(s+2)^2 + B(s+1)(s+2) + C(s+1) = 8 - (s+2)(4s+10). \tag{8.2.12}$$

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all s if they are equal for any three distinct values of s. We may determine A, B and C by choosing convenient values of s.

The left side of (8.2.12) suggests that we take $s = -2$ to obtain $C = -8$, and $s = -1$ to obtain $A = 2$. We can now choose any third value of s to determine B. Taking $s = 0$ yields $4A + 2B + C = -12$. Since $A = 2$ and $C = -8$ this implies that $B = -6$. Therefore

$$F(s) = \frac{2}{s+1} - \frac{6}{s+2} - \frac{8}{(s+2)^2}$$

and

$$\begin{aligned} L^{-1}(F) &= 2L^{-1}\left(\frac{1}{s+1}\right) - 6L^{-1}\left(\frac{1}{s+2}\right) - 8L^{-1}\left(\frac{1}{(s+2)^2}\right) \\ &= 2e^{-t} - 6e^{-2t} - 8te^{-2t}. \end{aligned}$$

Example 8.2.7 Find the inverse Laplace transform of

$$F(s) = \frac{s^2 - 5s + 7}{(s+2)^3}.$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}.$$

The easiest way to obtain A, B, and C is to expand the numerator in powers of $s + 2$. This yields

$$s^2 - 5s + 7 = [(s+2) - 2]^2 - 5[(s+2) - 2] + 7 = (s+2)^2 - 9(s+2) + 21.$$

Therefore

$$\begin{aligned} F(s) &= \frac{(s+2)^2 - 9(s+2) + 21}{(s+2)^3} \\ &= \frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3} \end{aligned}$$

and

$$\begin{aligned} L^{-1}(F) &= L^{-1}\left(\frac{1}{s+2}\right) - 9L^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{21}{2}L^{-1}\left(\frac{2}{(s+2)^3}\right) \\ &= e^{-2t}\left(1 - 9t + \frac{21}{2}t^2\right). \end{aligned}$$

Example 8.2.8 Find the inverse Laplace transform of

$$F(s) = \frac{1 - s(5+3s)}{s\left[(s+1)^2 + 1\right]}. \tag{8.2.13}$$

Solution One form for the partial fraction expansion of F is

$$F(s) = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 1}. \tag{8.2.14}$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of (8.2.14) will be a linear combination of the inverse transforms

$$e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t$$

of

$$\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}$$

respectively. Therefore, instead of (8.2.14) we write

$$F(s) = \frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2 + 1}. \tag{8.2.15}$$

Finding a common denominator yields

$$F(s) = \frac{A\left[(s+1)^2 + 1\right] + B(s+1)s + Cs}{s\left[(s+1)^2 + 1\right]}. \tag{8.2.16}$$

If (8.2.13) and (8.2.16) are to be equivalent, then

$$A\left[(s+1)^2 + 1\right] + B(s+1)s + Cs = 1 - s(5 + 3s).$$

This is true for all s if it's true for three distinct values of s. Choosing $s = 0, -1$, and 1 yields the system

$$\begin{aligned} 2A &= 1 \\ A - C &= 3 \\ 5A + 2B + C &= -7. \end{aligned}$$

Solving this system yields

$$A = \frac{1}{2}, \quad B = -\frac{7}{2}, \quad C = -\frac{5}{2}.$$

Hence, from (8.2.15),

$$F(s) = \frac{1}{2s} - \frac{7}{2}\frac{s+1}{(s+1)^2+1} - \frac{5}{2}\frac{1}{(s+1)^2+1}.$$

Therefore

$$\begin{aligned} L^{-1}(F) &= \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - \frac{7}{2}L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{5}{2}L^{-1}\left(\frac{1}{(s+1)^2+1}\right) \\ &= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t. \end{aligned}$$

Example 8.2.9 Find the inverse Laplace transform of

$$F(s) = \frac{8 + 3s}{(s^2+1)(s^2+4)}. \tag{8.2.17}$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A + Bs}{s^2 + 1} + \frac{C + Ds}{s^2 + 4}.$$

The coefficients A, B, C and D can be obtained by finding a common denominator and equating the resulting numerator to the numerator in (8.2.17). However, since there's no first power of s in the denominator of (8.2.17), there's an easier way: the expansion of

$$F_1(s) = \frac{1}{(s^2 + 1)(s^2 + 4)}$$

can be obtained quickly by using Heaviside's method to expand

$$\frac{1}{(x + 1)(x + 4)} = \frac{1}{3}\left(\frac{1}{x + 1} - \frac{1}{x + 4}\right)$$

and then setting $x = s^2$ to obtain

$$\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3}\left(\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4}\right).$$

Multiplying this by $8 + 3s$ yields

$$F(s) = \frac{8 + 3s}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3}\left(\frac{8 + 3s}{s^2 + 1} - \frac{8 + 3s}{s^2 + 4}\right).$$

Therefore

$$L^{-1}(F) = \frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t.$$

USING TECHNOLOGY

Some software packages that do symbolic algebra can find partial fraction expansions very easily. We recommend that you use such a package if one is available to you, but only after you've done enough partial fraction expansions on your own to master the technique.

8.2 Exercises

1. Use the table of Laplace transforms to find the inverse Laplace transform.

(a) $\frac{3}{(s-7)^4}$ **(b)** $\frac{2s-4}{s^2-4s+13}$ **(c)** $\frac{1}{s^2+4s+20}$

(d) $\frac{2}{s^2+9}$ **(e)** $\frac{s^2-1}{(s^2+1)^2}$ **(f)** $\frac{1}{(s-2)^2-4}$

(g) $\frac{12s-24}{(s^2-4s+85)^2}$ **(h)** $\frac{2}{(s-3)^2-9}$ **(i)** $\frac{s^2-4s+3}{(s^2-4s+5)^2}$

2. Use Theorem 8.2.1 and the table of Laplace transforms to find the inverse Laplace transform.

(a) $\dfrac{2s+3}{(s-7)^4}$ **(b)** $\dfrac{s^2-1}{(s-2)^6}$ **(c)** $\dfrac{s+5}{s^2+6s+18}$

(d) $\dfrac{2s+1}{s^2+9}$ **(e)** $\dfrac{s}{s^2+2s+1}$ **(f)** $\dfrac{s+1}{s^2-9}$

(g) $\dfrac{s^3+2s^2-s-3}{(s+1)^4}$ **(h)** $\dfrac{2s+3}{(s-1)^2+4}$ **(i)** $\dfrac{1}{s}-\dfrac{s}{s^2+1}$

(j) $\dfrac{3s+4}{s^2-1}$ **(k)** $\dfrac{3}{s-1}+\dfrac{4s+1}{s^2+9}$ **(l)** $\dfrac{3}{(s+2)^2}-\dfrac{2s+6}{s^2+4}$

3. Use Heaviside's method to find the inverse Laplace transform.

(a) $\dfrac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$ **(b)** $\dfrac{7+(s+4)(18-3s)}{(s-3)(s-1)(s+4)}$

(c) $\dfrac{2+(s-2)(3-2s)}{(s-2)(s+2)(s-3)}$ **(d)** $\dfrac{3-(s-1)(s+1)}{(s+4)(s-2)(s-1)}$

(e) $\dfrac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$ **(f)** $\dfrac{3+(s-3)(2s^2+s-21)}{(s-3)(s-1)(s+4)(s-2)}$

4. Find the inverse Laplace transform.

(a) $\dfrac{2+3s}{(s^2+1)(s+2)(s+1)}$ **(b)** $\dfrac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)}$

(c) $\dfrac{3s+2}{(s-2)(s^2+2s+5)}$ **(d)** $\dfrac{3s^2+2s+1}{(s-1)^2(s+2)(s+3)}$

(e) $\dfrac{2s^2+s+3}{(s-1)^2(s+2)^2}$ **(f)** $\dfrac{3s+2}{(s^2+1)(s-1)^2}$

5. Use the method of Example 8.2.9 to find the inverse Laplace transform.

(a) $\dfrac{3s+2}{(s^2+4)(s^2+9)}$ **(b)** $\dfrac{-4s+1}{(s^2+1)(s^2+16)}$ **(c)** $\dfrac{5s+3}{(s^2+1)(s^2+4)}$

(d) $\dfrac{-s+1}{(4s^2+1)(s^2+1)}$ **(e)** $\dfrac{17s-34}{(s^2+16)(16s^2+1)}$ **(f)** $\dfrac{2s-1}{(4s^2+1)(9s^2+1)}$

6. Find the inverse Laplace transform.

(a) $\dfrac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$ **(b)** $\dfrac{8s+56}{(s^2-6s+13)(s^2+2s+5)}$

(c) $\dfrac{s+9}{(s^2+4s+5)(s^2-4s+13)}$ **(d)** $\dfrac{3s-2}{(s^2-4s+5)(s^2-6s+13)}$

(e) $\dfrac{3s-1}{(s^2-2s+2)(s^2+2s+5)}$ **(f)** $\dfrac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)}$

7. Find the inverse Laplace transform.

(a) $\dfrac{1}{s(s^2+1)}$ **(b)** $\dfrac{1}{(s-1)(s^2-2s+17)}$

(c) $\dfrac{3s+2}{(s-2)(s^2+2s+10)}$ **(d)** $\dfrac{34-17s}{(2s-1)(s^2-2s+5)}$

(e) $\dfrac{s+2}{(s-3)(s^2+2s+5)}$ **(f)** $\dfrac{2s-2}{(s-2)(s^2+2s+10)}$

8. Find the inverse Laplace transform.

(a) $\dfrac{2s+1}{(s^2+1)(s-1)(s-3)}$ **(b)** $\dfrac{s+2}{(s^2+2s+2)(s^2-1)}$

(c) $\dfrac{2s-1}{(s^2-2s+2)(s+1)(s-2)}$ **(d)** $\dfrac{s-6}{(s^2-1)(s^2+4)}$

(e) $\dfrac{2s-3}{s(s-2)(s^2-2s+5)}$ **(f)** $\dfrac{5s-15}{(s^2-4s+13)(s-2)(s-1)}$

9. Given that $f(t) \leftrightarrow F(s)$, find the inverse Laplace transform of $F(as-b)$, where $a>0$.

10. **(a)** If $s_1, s_2, \dots, s_n$ are distinct and P is a polynomial of degree less than n, then

$$\frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)} = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n}.$$

Multiply through by $s-s_i$ to show that A_i can be obtained by ignoring the factor $s-s_i$ on the left and setting $s=s_i$ elsewhere.

(b) Suppose P and Q_1 are polynomials such that degree$(P) \le$ degree(Q_1) and $Q_1(s_1) \ne 0$. Show that the coefficient of $1/(s-s_1)$ in the partial fraction expansion of

$$F(s) = \frac{P(s)}{(s-s_1)Q_1(s)}$$

is $P(s_1)/Q_1(s_1)$.

(c) Explain how the results of **(a)** and **(b)** are related.

8.3 SOLUTION OF INITIAL VALUE PROBLEMS

Laplace Transforms of Derivatives

In the rest of this chapter we'll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of f' is related to the Laplace transform of f. The next theorem answers this question.

Theorem 8.3.1 *Suppose f is continuous on $[0,\infty)$ and of exponential order s_0, and f' is piecewise continuous on $[0,\infty)$. Then f and f' have Laplace transforms for $s>s_0$, and*

$$L(f') = sL(f) - f(0). \tag{8.3.1}$$

Proof

We know from Theorem 8.1.6 that $L(f)$ is defined for $s>s_0$. We first consider the case where f' is continuous on $[0,\infty)$. Integration by parts yields

$$\begin{aligned} \int_0^T e^{-st} f'(t)\,dt &= e^{-st} f(t)\Big|_0^T + s\int_0^T e^{-st} f(t)\,dt \\ &= e^{-sT} f(T) - f(0) + s\int_0^T e^{-st} f(t)\,dt \end{aligned} \tag{8.3.2}$$

for any $T>0$. Since f is of exponential order s_0, $\lim_{T\to\infty} e^{-sT} f(T) = 0$ and the last integral in (8.3.2) converges as $T\to\infty$ if $s>s_0$. Therefore

$$\begin{aligned} \int_0^\infty e^{-st} f'(t)\,dt &= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt \\ &= -f(0) + sL(f), \end{aligned}$$

which proves (8.3.1). Now suppose $T > 0$ and f' is only piecewise continuous on $[0, T]$, with discontinuities at $t_1 < t_2 < \cdots < t_{n-1}$. For convenience, let $t_0 = 0$ and $t_n = T$. Integrating by parts yields

$$\begin{aligned}\int_{t_{i-1}}^{t_i} e^{-st} f'(t)\,dt &= e^{-st} f(t)\Big|_{t_{i-1}}^{t_i} + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\,dt \\ &= e^{-st_i} f(t_i) - e^{-st_{i-1}} f(t_{i-1}) + s\int_{t_{i-1}}^{t_i} e^{-st} f(t)\,dt.\end{aligned}$$

Summing both sides of this equation from $i = 1$ to n and noting that

$$\begin{aligned}&\left(e^{-st_1} f(t_1) - e^{-st_0} f(t_0)\right) + \left(e^{-st_2} f(t_2) - e^{-st_1} f(t_1)\right) + \cdots + \left(e^{-st_N} f(t_N) - e^{-st_{N-1}} f(t_{N-1})\right) \\ &\qquad = e^{-st_N} f(t_N) - e^{-st_0} f(t_0) = e^{-sT} f(T) - f(0)\end{aligned}$$

yields (8.3.2), so (8.3.1) follows as before.

Example 8.3.1 In Example 8.1.4 we saw that

$$L(\cos\omega t) = \frac{s}{s^2+\omega^2}.$$

Applying (8.3.1) with $f(t) = \cos\omega t$ shows that

$$L(-\omega\sin\omega t) = s\frac{s}{s^2+\omega^2} - 1 = -\frac{\omega^2}{s^2+\omega^2}.$$

Therefore

$$L(\sin\omega t) = \frac{\omega}{s^2+\omega^2},$$

which agrees with the corresponding result obtained in 8.1.4.

In Section 2.1 we showed that the solution of the initial value problem

$$y' = ay, \quad y(0) = y_0, \tag{8.3.3}$$

is $y = y_0e^{at}$. We'll now obtain this result by using the Laplace transform.

Let $Y(s) = L(y)$ be the Laplace transform of the unknown solution of (8.3.3). Taking Laplace transforms of both sides of (8.3.3) yields

$$L(y') = L(ay),$$

which, by Theorem 8.3.1, can be rewritten as

$$sL(y) - y(0) = aL(y),$$

or

$$sY(s) - y_0 = aY(s).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{y_0}{s-a},$$

so

$$y = L^{-1}(Y(s)) = L^{-1}\left(\frac{y_0}{s-a}\right) = y_0L^{-1}\left(\frac{1}{s-a}\right) = y_0e^{at},$$

which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

Theorem 8.3.2 *Suppose f and f' are continuous on $[0,\infty)$ and of exponential order s_0, and that f'' is piecewise continuous on $[0,\infty)$. Then f, f', and f'' have Laplace transforms for $s > s_0$,*

$$L(f') = sL(f) - f(0), \tag{8.3.4}$$

and

$$L(f'') = s^2L(f) - f'(0) - sf(0). \tag{8.3.5}$$

Proof Theorem 8.3.1 implies that $L(f')$ exists and satisfies (8.3.4) for $s > s_0$. To prove that $L(f'')$ exists and satisfies (8.3.5) for $s > s_0$, we first apply Theorem 8.3.1 to $g = f'$. Since g satisfies the hypotheses of Theorem 8.3.1, we conclude that $L(g')$ is defined and satisfies

$$L(g') = sL(g) - g(0)$$

for $s > s_0$. However, since $g' = f''$, this can be rewritten as

$$L(f'') = sL(f') - f'(0).$$

Substituting (8.3.4) into this yields (8.3.5).

Solving Second Order Equations with the Laplace Transform

We'll now use the Laplace transform to solve initial value problems for second order equations.

Example 8.3.2 Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3. \tag{8.3.6}$$

Solution Taking Laplace transforms of both sides of the differential equation in (8.3.6) yields

$$L(y'' - 6y' + 5y) = L\left(3e^{2t}\right) = \frac{3}{s-2},$$

which we rewrite as

$$L(y'') - 6L(y') + 5L(y) = \frac{3}{s-2}. \tag{8.3.7}$$

Now denote $L(y) = Y(s)$. Theorem 8.3.2 and the initial conditions in (8.3.6) imply that

$$L(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$L(y'') = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 3 - 2s.$$

Substituting from the last two equations into (8.3.7) yields

$$\left(s^2Y(s) - 3 - 2s\right) - 6\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (3 + 2s) + 6(-2), \tag{8.3.8}$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-5} + \frac{5}{2}\frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}$$

as the solution of (8.3.6).

It isn't necessary to write all the steps that we used to obtain (8.3.8). To see how to avoid this, let's apply the method of Example 8.3.2 to the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{8.3.9}$$

Taking Laplace transforms of both sides of the differential equation in (8.3.9) yields

$$aL(y'') + bL(y') + cL(y) = F(s). \tag{8.3.10}$$

Now let $Y(s) = L(y)$. Theorem 8.3.2 and the initial conditions in (8.3.9) imply that

$$L(y') = sY(s) - k_0 \quad \text{and} \quad L(y'') = s^2Y(s) - k_1 - k_0 s.$$

Substituting these into (8.3.10) yields

$$a\left(s^2Y(s) - k_1 - k_0 s\right) + b\left(sY(s) - k_0\right) + cY(s) = F(s). \tag{8.3.11}$$

The coefficient of $Y(s)$ on the left is the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation for (8.3.9). Using this and moving the terms involving k_0 and k_1 to the right side of (8.3.11) yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0. \tag{8.3.12}$$

This equation corresponds to (8.3.8) of Example 8.3.2. Having established the form of this equation in the general case, it is preferable to go directly from the initial value problem to this equation. You may find it easier to remember (8.3.12) rewritten as

$$p(s)Y(s) = F(s) + a\left(y'(0) + sy(0)\right) + by(0). \tag{8.3.13}$$

Example 8.3.3 Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4, \; y'(0) = 2. \tag{8.3.14}$$

Solution The characteristic polynomial is

$$p(s) = 2s^2 + 3s + 1 = (2s+1)(s+1)$$

and

$$F(s) = L(8e^{-2t}) = \frac{8}{s+2},$$

so (8.3.13) becomes

$$(2s+1)(s+1)Y(s) = \frac{8}{s+2} + 2(2-4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4\,(1-(s+2)(s+1))}{(s+1/2)(s+1)(s+2)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = \frac{4}{3}\frac{1}{s+1/2} - \frac{8}{s+1} + \frac{8}{3}\frac{1}{s+2},$$

so the solution of (8.3.14) is

$$y = L^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 8.3.1).

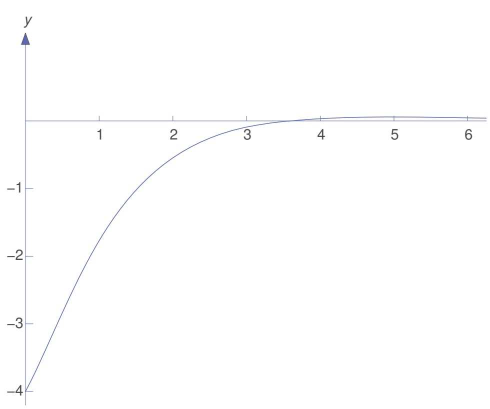

Figure 8.3.1 $y = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$

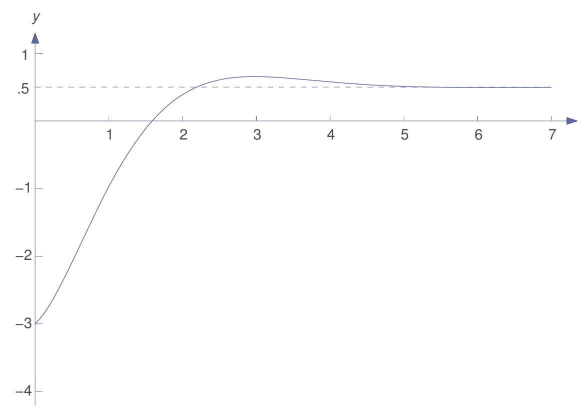

Figure 8.3.2 $y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$

Example 8.3.4 Solve the initial value problem

$$y'' + 2y' + 2y = 1, \quad y(0) = -3, \; y'(0) = 1. \tag{8.3.15}$$

Solution The characteristic polynomial is

$$p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$$

and

$$F(s) = L(1) = \frac{1}{s},$$

so (8.3.13) becomes

$$\left[(s+1)^2+1\right]Y(s)=\frac{1}{s}+1\cdot(1-3s)+2(-3).$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{1-s(5+3s)}{s\left[(s+1)^2+1\right]}.$$

In Example 8.2.8 we found the inverse transform of this function to be

$$y=\frac{1}{2}-\frac{7}{2}e^{-t}\cos t-\frac{5}{2}e^{-t}\sin t$$

(Figure 8.3.2), which is therefore the solution of (8.3.15).

REMARK: In our examples we applied Theorems 8.3.1 and 8.3.2 without verifying that the unknown function y satisfies their hypotheses. This is characteristic of the formal manipulative way in which the Laplace transform is used to solve differential equations. Any doubts about the validity of the method for solving a given equation can be resolved by verifying that the resulting function y is the solution of the given problem.

8.3 Exercises

In Exercises 1–31 use the Laplace transform to solve the initial value problem.

1. $y''+3y'+2y=e^t,\quad y(0)=1,\quad y'(0)=-6$

2. $y''-y'-6y=2,\quad y(0)=1,\quad y'(0)=0$

3. $y''+y'-2y=2e^{3t},\quad y(0)=-1,\quad y'(0)=4$

4. $y''-4y=2e^{3t},\quad y(0)=1,\quad y'(0)=-1$

5. $y''+y'-2y=e^{3t},\quad y(0)=1,\quad y'(0)=-1$

6. $y''+3y'+2y=6e^t,\quad y(0)=1,\quad y'(0)=-1$

7. $y''+y=\sin 2t,\quad y(0)=0,\quad y'(0)=1$

8. $y''-3y'+2y=2e^{3t},\quad y(0)=1,\quad y'(0)=-1$

9. $y''-3y'+2y=e^{4t},\quad y(0)=1,\quad y'(0)=-2$

10. $y''-3y'+2y=e^{3t},\quad y(0)=-1,\quad y'(0)=-4$

11. $y''+3y'+2y=2e^t,\quad y(0)=0,\quad y'(0)=-1$

12. $y''+y'-2y=-4,\quad y(0)=2,\quad y'(0)=3$

13. $y''+4y=4,\quad y(0)=0,\quad y'(0)=1$

14. $y''-y'-6y=2,\quad y(0)=1,\quad y'(0)=0$

15. $y''+3y'+2y=e^t,\quad y(0)=0,\quad y'(0)=1$

16. $y''-y=1,\quad y(0)=1,\quad y'(0)=0$

17. $y''+4y=3\sin t,\quad y(0)=1,\quad y'(0)=-1$

18. $y''+y'=2e^{3t},\quad y(0)=-1,\quad y'(0)=4$

19. $y''+y=1,\quad y(0)=2,\quad y'(0)=0$

20. $y''+y=t,\quad y(0)=0,\quad y'(0)=2$

21. $y'' + y = t - 3\sin 2t, \quad y(0) = 1, \quad y'(0) = -3$

22. $y'' + 5y' + 6y = 2e^{-t}, \quad y(0) = 1, \quad y'(0) = 3$

23. $y'' + 2y' + y = 6\sin t - 4\cos t, \quad y(0) = -1,\ y'(0) = 1$

24. $y'' - 2y' - 3y = 10\cos t, \quad y(0) = 2, \quad y'(0) = 7$

25. $y'' + y = 4\sin t + 6\cos t, \quad y(0) = -6,\ y'(0) = 2$

26. $y'' + 4y = 8\sin 2t + 9\cos t, \quad y(0) = 1, \quad y'(0) = 0$

27. $y'' - 5y' + 6y = 10e^{t}\cos t, \quad y(0) = 2, \quad y'(0) = 1$

28. $y'' + 2y' + 2y = 2t, \quad y(0) = 2, \quad y'(0) = -7$

29. $y'' - 2y' + 2y = 5\sin t + 10\cos t, \quad y(0) = 1,\ y'(0) = 2$

30. $y'' + 4y' + 13y = 10e^{-t} - 36e^{t}, \quad y(0) = 0,\ y'(0) = -16$

31. $y'' + 4y' + 5y = e^{-t}(\cos t + 3\sin t), \quad y(0) = 0, \quad y'(0) = 4$

32. $2y'' - 3y' - 2y = 4e^{t}, \quad y(0) = 1,\ y'(0) = -2$

33. $6y'' - y' - y = 3e^{2t}, \quad y(0) = 0,\ y'(0) = 0$

34. $2y'' + 2y' + y = 2t, \quad y(0) = 1,\ y'(0) = -1$

35. $4y'' - 4y' + 5y = 4\sin t - 4\cos t, \quad y(0) = 0,\ y'(0) = 11/17$

36. $4y'' + 4y' + y = 3\sin t + \cos t, \quad y(0) = 2,\ y'(0) = -1$

37. $9y'' + 6y' + y = 3e^{3t}, \quad y(0) = 0,\ y'(0) = -3$

38. Suppose a, b, and c are constants and $a \neq 0$. Let

$$y_1 = L^{-1}\left(\frac{as + b}{as^2 + bs + c}\right) \quad \text{and} \quad y_2 = L^{-1}\left(\frac{a}{as^2 + bs + c}\right).$$

Show that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

HINT: *Use the Laplace transform to solve the initial value problems*

$$\begin{aligned} ay'' + by' + cy &= 0, \quad y(0) = 1, \quad y'(0) = 0 \\ ay'' + by' + cy &= 0, \quad y(0) = 0, \quad y'(0) = 1. \end{aligned}$$

8.4 THE UNIT STEP FUNCTION

In the next section we'll consider initial value problems

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where a, b, and c are constants and f is piecewise continuous. In this section we'll develop procedures for using the table of Laplace transforms to find Laplace transforms of piecewise continuous functions, and to find the piecewise continuous inverses of Laplace transforms.

Example 8.4.1 Use the table of Laplace transforms to find the Laplace transform of

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 2, \\ 3t, & t \ge 2 \end{cases} \tag{8.4.1}$$

(Figure 8.4.1).

Solution Since the formula for f changes at $t = 2$, we write

$$\begin{aligned} L(f) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^2 e^{-st}(2t+1)\,dt + \int_2^\infty e^{-st}(3t)\,dt. \end{aligned} \tag{8.4.2}$$

To relate the first term to a Laplace transform, we add and subtract

$$\int_2^\infty e^{-st}(2t+1)\,dt$$

in (8.4.2) to obtain

$$\begin{aligned} L(f) &= \int_0^\infty e^{-st}(2t+1)\,dt + \int_2^\infty e^{-st}(3t-2t-1)\,dt \\ &= \int_0^\infty e^{-st}(2t+1)\,dt + \int_2^\infty e^{-st}(t-1)\,dt \\ &= L(2t+1) + \int_2^\infty e^{-st}(t-1)\,dt. \end{aligned} \tag{8.4.3}$$

To relate the last integral to a Laplace transform, we make the change of variable $x = t - 2$ and rewrite the integral as

$$\begin{aligned} \int_2^\infty e^{-st}(t-1)\,dt &= \int_0^\infty e^{-s(x+2)}(x+1)\,dx \\ &= e^{-2s}\int_0^\infty e^{-sx}(x+1)\,dx. \end{aligned}$$

Since the symbol used for the variable of integration has no effect on the value of a definite integral, we can now replace x by the more standard t and write

$$\int_2^\infty e^{-st}(t-1)\,dt = e^{-2s}\int_0^\infty e^{-st}(t+1)\,dt = e^{-2s}L(t+1).$$

This and (8.4.3) imply that

$$L(f) = L(2t+1) + e^{-2s}L(t+1).$$

Now we can use the table of Laplace transforms to find that

$$L(f) = \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

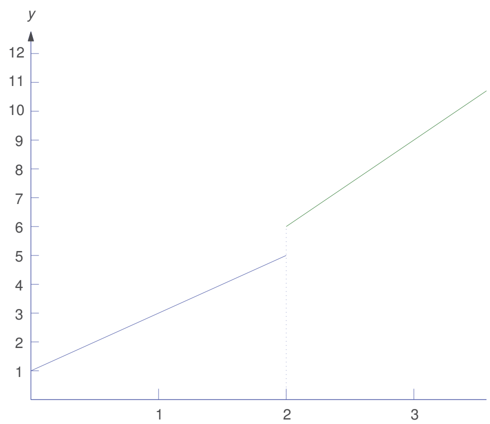

Figure 8.4.1 The piecewise continuous function (8.4.1)

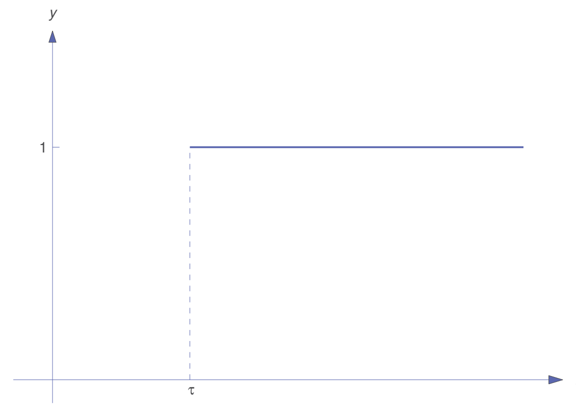

Figure 8.4.2 $y = u(t - \tau)$

Laplace Transforms of Piecewise Continuous Functions

We'll now develop the method of Example 8.4.1 into a systematic way to find the Laplace transform of a piecewise continuous function. It is convenient to introduce the *unit step function*, defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \tag{8.4.4}$$

Thus, $u(t)$ "steps" from the constant value 0 to the constant value 1 at $t = 0$. If we replace t by $t - \tau$ in (8.4.4), then

$$u(t-\tau) = \begin{cases} 0, & t < \tau, \\ 1, & t \geq \tau \end{cases};$$

that is, the step now occurs at $t = \tau$ (Figure 8.4.2).

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1, \end{cases} \tag{8.4.5}$$

where we assume that f_0 and f_1 are defined on $[0, \infty)$, even though they equal f only on the indicated intervals. This assumption enables us to rewrite (8.4.5) as

$$f(t) = f_0(t) + u(t - t_1)\left(f_1(t) - f_0(t)\right). \tag{8.4.6}$$

To verify this, note that if $t < t_1$ then $u(t - t_1) = 0$ and (8.4.6) becomes

$$f(t) = f_0(t) + (0)\left(f_1(t) - f_0(t)\right) = f_0(t).$$

If $t \geq t_1$ then $u(t - t_1) = 1$ and (8.4.6) becomes

$$f(t) = f_0(t) + (1)\left(f_1(t) - f_0(t)\right) = f_1(t).$$

We need the next theorem to show how (8.4.6) can be used to find $L(f)$.

Theorem 8.4.1 *Let g be defined on $[0,\infty)$. Suppose $\tau \ge 0$ and $L\left(g(t+\tau)\right)$ exists for $s > s_0$. Then $L\left(u(t-\tau)g(t)\right)$ exists for $s > s_0$, and*

$$L(u(t-\tau)g(t)) = e^{-s\tau} L\left(g(t+\tau)\right).$$

Proof By definition,

$$L\left(u(t-\tau)g(t)\right) = \int_0^\infty e^{-st} u(t-\tau)g(t)\,dt.$$

From this and the definition of $u(t-\tau)$,

$$L\left(u(t-\tau)g(t)\right) = \int_0^\tau e^{-st}(0)\,dt + \int_\tau^\infty e^{-st} g(t)\,dt.$$

The first integral on the right equals zero. Introducing the new variable of integration $x = t - \tau$ in the second integral yields

$$L\left(u(t-\tau)g(t)\right) = \int_0^\infty e^{-s(x+\tau)} g(x+\tau)\,dx = e^{-s\tau}\int_0^\infty e^{-sx} g(x+\tau)\,dx.$$

Changing the name of the variable of integration in the last integral from x to t yields

$$L\left(u(t-\tau)g(t)\right) = e^{-s\tau}\int_0^\infty e^{-st} g(t+\tau)\,dt = e^{-s\tau} L(g(t+\tau)). \blacksquare$$

Example 8.4.2 Find

$$L\left(u(t-1)(t^2+1)\right).$$

Solution Here $\tau = 1$ and $g(t) = t^2 + 1$, so

$$g(t+1) = (t+1)^2 + 1 = t^2 + 2t + 2.$$

Since

$$L\left(g(t+1)\right) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s},$$

Theorem 8.4.1 implies that

$$L\left(u(t-1)(t^2+1)\right) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right).$$

Example 8.4.3 Use Theorem 8.4.1 to find the Laplace transform of the function

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 2, \\ 3t, & t \ge 2, \end{cases}$$

from Example 8.4.1.

Solution We first write f in the form (8.4.6) as

$$f(t) = 2t + 1 + u(t-2)(t-1).$$

Therefore

$$\begin{aligned} L(f) &= L(2t+1) + L\left(u(t-2)(t-1)\right) \\ &= L(2t+1) + e^{-2s}L(t+1) \quad \text{(from Theorem 8.4.1)} \\ &= \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right), \end{aligned}$$

which is the result obtained in Example 8.4.1.

Formula (8.4.6) can be extended to more general piecewise continuous functions. For example, we can write

$$f(t) = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t_1 \le t < t_2, \\ f_2(t), & t \ge t_2, \end{cases}$$

as

$$f(t) = f_0(t) + u(t-t_1)\left(f_1(t) - f_0(t)\right) + u(t-t_2)\left(f_2(t) - f_1(t)\right)$$

if f_0, f_1, and f_2 are all defined on $[0, \infty)$.

Example 8.4.4 Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ -2t+1, & 2 \le t < 3, \\ 3t, & 3 \le t < 5, \\ t-1, & t \ge 5 \end{cases} \tag{8.4.7}$$

(Figure 8.4.3).

Solution In terms of step functions,

$$\begin{aligned} f(t) &= 1 + u(t-2)(-2t+1-1) + u(t-3)(3t+2t-1) \\ &\quad + u(t-5)(t-1-3t), \end{aligned}$$

or

$$f(t) = 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned} L(f) &= L(1) - 2e^{-2s}L(t+2) + e^{-3s}L\left(5(t+3)-1\right) - e^{-5s}L\left(2(t+5)+1\right) \\ &= L(1) - 2e^{-2s}L(t+2) + e^{-3s}L(5t+14) - e^{-5s}L(2t+11) \\ &= \frac{1}{s} - 2e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2} + \frac{14}{s}\right) - e^{-5s}\left(\frac{2}{s^2} + \frac{11}{s}\right). \end{aligned}$$

The trigonometric identities

$$\sin(A+B) = \sin A\cos B + \cos A \sin B \tag{8.4.8}$$

$$\cos(A+B) = \cos A\cos B - \sin A \sin B \tag{8.4.9}$$

are useful in problems that involve shifting the arguments of trigonometric functions. We'll use these identities in the next example.

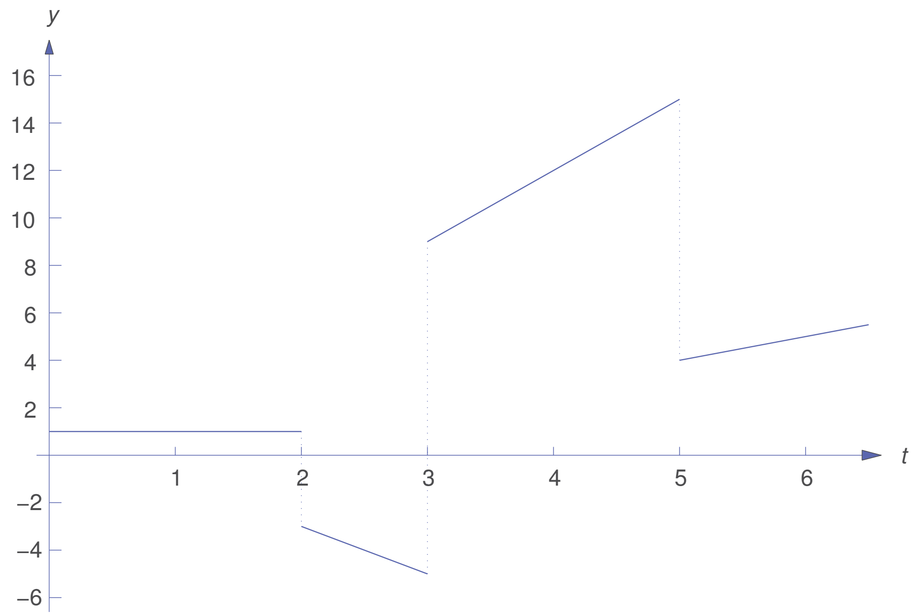

Figure 8.4.3 The piecewise contnuous function (8.4.7)

Example 8.4.5 Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \le t < \dfrac{\pi}{2}, \\ \cos t - 3\sin t, & \dfrac{\pi}{2} \le t < \pi, \\ 3\cos t, & t \ge \pi \end{cases} \tag{8.4.10}$$

(Figure 8.4.4).

Solution In terms of step functions,

$$f(t) = \sin t + u(t - \pi/2)(\cos t - 4\sin t) + u(t - \pi)(2\cos t + 3\sin t).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned} L(f) &= L(\sin t) + e^{-\frac{\pi}{2}s} L\left(\cos\left(t + \tfrac{\pi}{2}\right) - 4\sin\left(t + \tfrac{\pi}{2}\right)\right) \\ &\quad + e^{-\pi s} L\left(2\cos(t + \pi) + 3\sin(t + \pi)\right). \end{aligned} \tag{8.4.11}$$

Since

$$\cos\left(t + \frac{\pi}{2}\right) - 4\sin\left(t + \frac{\pi}{2}\right) = -\sin t - 4\cos t$$

and

$$2\cos(t + \pi) + 3\sin(t + \pi) = -2\cos t - 3\sin t,$$

we see from (8.4.11) that

$$\begin{aligned} L(f) &= L(\sin t) - e^{-\pi s/2} L(\sin t + 4\cos t) - e^{-\pi s} L(2\cos t + 3\sin t) \\ &= \frac{1}{s^2 + 1} - e^{-\frac{\pi}{2}s}\left(\frac{1 + 4s}{s^2 + 1}\right) - e^{-\pi s}\left(\frac{3 + 2s}{s^2 + 1}\right). \end{aligned}$$

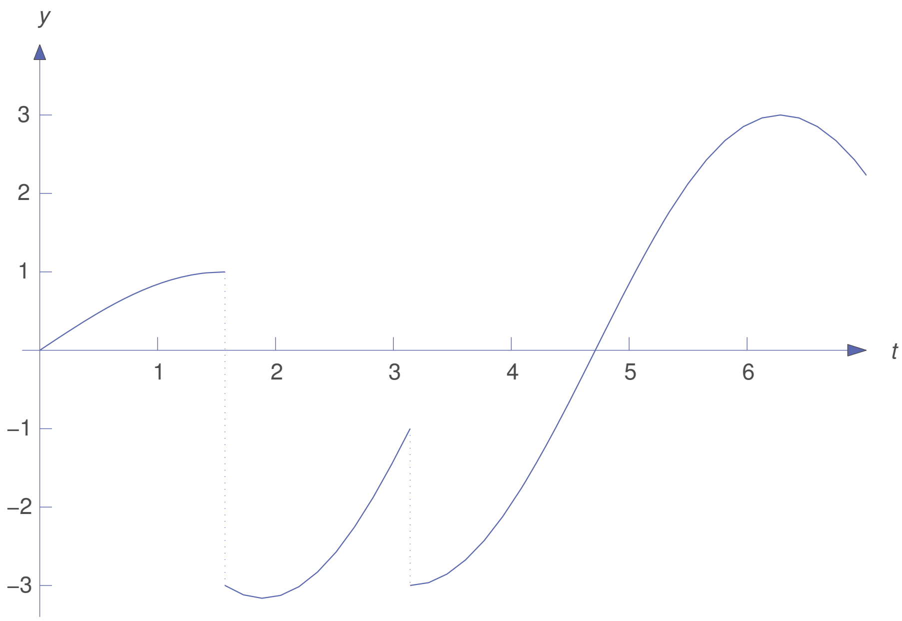

Figure 8.4.4 The piecewise continuous function (8.4.10)

The Second Shifting Theorem

Replacing $g(t)$ by $g(t-\tau)$ in Theorem 8.4.1 yields the next theorem.

Theorem 8.4.2 *[Second Shifting Theorem] If $\tau \geq 0$ and $L(g)$ exists for $s > s_0$ then $L\left(u(t-\tau)g(t-\tau)\right)$ exists for $s > s_0$ and*

$$L(u(t-\tau)g(t-\tau)) = e^{-s\tau} L(g(t)),$$

or, equivalently,

$$\text{if } g(t) \leftrightarrow G(s), \text{ then } u(t-\tau)g(t-\tau) \leftrightarrow e^{-s\tau} G(s). \tag{8.4.12}$$

REMARK: Recall that the First Shifting Theorem (Theorem 8.1.3 states that multiplying a function by e^{at} corresponds to shifting the argument of its transform by a units. Theorem 8.4.2 states that multiplying a Laplace transform by the exponential $e^{-\tau s}$ corresponds to shifting the argument of the inverse transform by τ units.

Example 8.4.6 Use (8.4.12) to find

$$L^{-1}\left(\frac{e^{-2s}}{s^2}\right).$$

Solution To apply (8.4.12) we let $\tau = 2$ and $G(s) = 1/s^2$. Then $g(t) = t$ and (8.4.12) implies that

$$L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2)(t-2).$$

Example 8.4.7 Find the inverse Laplace transform h of

$$H(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-4s}\left(\frac{4}{s^3} + \frac{1}{s}\right),$$

and find distinct formulas for h on appropriate intervals.

Solution Let

$$G_0(s) = \frac{1}{s^2}, \quad G_1(s) = \frac{1}{s^2} + \frac{2}{s}, \quad G_2(s) = \frac{4}{s^3} + \frac{1}{s}.$$

Then

$$g_0(t) = t, \; g_1(t) = t + 2, \; g_2(t) = 2t^2 + 1.$$

Hence, (8.4.12) and the linearity of L^{-1} imply that

$$\begin{aligned} h(t) &= L^{-1}\left(G_0(s)\right) - L^{-1}\left(e^{-s}G_1(s)\right) + L^{-1}\left(e^{-4s}G_2(s)\right) \\ &= t - u(t-1)\left[(t-1)+2\right] + u(t-4)\left[2(t-4)^2+1\right] \\ &= t - u(t-1)(t+1) + u(t-4)(2t^2 - 16t + 33), \end{aligned}$$

which can also be written as

$$h(t) = \begin{cases} t, & 0 \le t < 1, \\ -1, & 1 \le t < 4, \\ 2t^2 - 16t + 32, & t \ge 4. \end{cases}$$

Example 8.4.8 Find the inverse transform of

$$H(s) = \frac{2s}{s^2+4} - e^{-\frac{\pi}{2}s}\frac{3s+1}{s^2+9} + e^{-\pi s}\frac{s+1}{s^2+6s+10}.$$

Solution Let

$$G_0(s) = \frac{2s}{s^2+4}, \quad G_1(s) = -\frac{(3s+1)}{s^2+9},$$

and

$$G_2(s) = \frac{s+1}{s^2+6s+10} = \frac{(s+3)-2}{(s+3)^2+1}.$$

Then

$$g_0(t) = 2\cos 2t, \quad g_1(t) = -3\cos 3t - \frac{1}{3}\sin 3t,$$

and

$$g_2(t) = e^{-3t}(\cos t - 2\sin t).$$

Therefore (8.4.12) and the linearity of L^{-1} imply that

$$\begin{aligned} h(t) &= 2\cos 2t - u(t-\pi/2)\left[3\cos 3(t-\pi/2) + \frac{1}{3}\sin 3\left(t - \frac{\pi}{2}\right)\right] \\ &\quad + u(t-\pi)e^{-3(t-\pi)}\left[\cos(t-\pi) - 2\sin(t-\pi)\right]. \end{aligned}$$

Using the trigonometric identities (8.4.8) and (8.4.9), we can rewrite this as

$$\begin{aligned} h(t) &= 2\cos 2t + u(t-\pi/2)\left(3\sin 3t - \tfrac{1}{3}\cos 3t\right) \\ &\quad -u(t-\pi)e^{-3(t-\pi)}(\cos t - 2\sin t) \end{aligned} \tag{8.4.13}$$

(Figure 8.4.5).

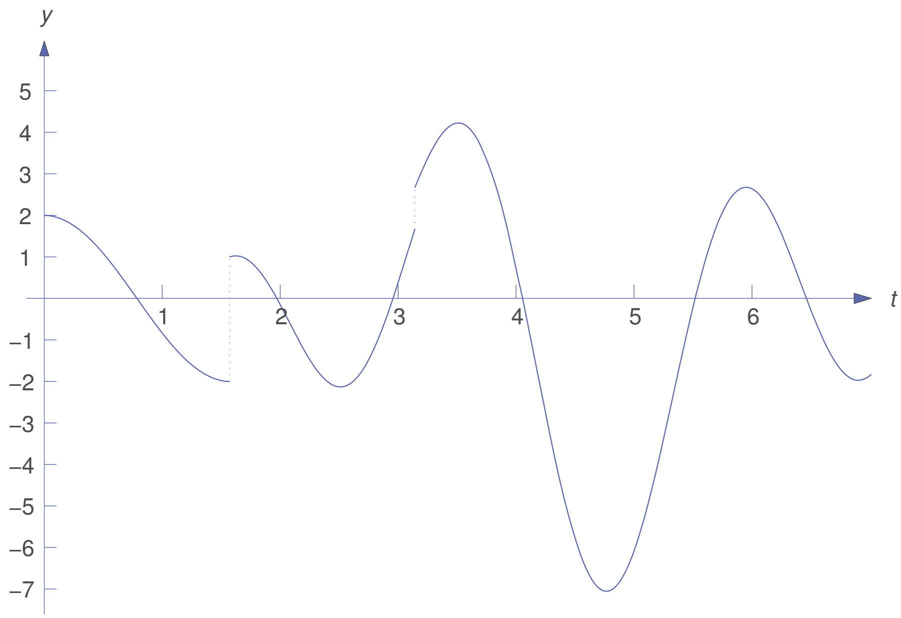

Figure 8.4.5 The piecewise continouous function (8.4.13)

8.4 Exercises

In Exercises 1–6 find the Laplace transform by the method of Example 8.4.1. Then express the given function f in terms of unit step functions as in Eqn. (8.4.6), and use Theorem 8.4.1 to find $L(f)$. Where indicated by C/G , graph f.

1. $f(t) = \begin{cases} 1, & 0 \le t < 4, \\ t, & t \ge 4. \end{cases}$

2. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$

3. C/G $f(t) = \begin{cases} 2t-1, & 0 \le t < 2, \\ t, & t \ge 2. \end{cases}$

4. C/G $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ t+2, & t \ge 1. \end{cases}$

5. $f(t) = \begin{cases} t-1, & 0 \le t < 2, \\ 4, & t \ge 2. \end{cases}$

6. $f(t) = \begin{cases} t^2, & 0 \le t < 1, \\ 0, & t \ge 1. \end{cases}$

In Exercises 7–18 express the given function f in terms of unit step functions and use Theorem 8.4.1 to find $L(f)$. Where indicated by C/G *, graph f.*

7. $f(t) = \begin{cases} 0, & 0 \le t < 2, \\ t^2 + 3t, & t \ge 2. \end{cases}$

8. $f(t) = \begin{cases} t^2 + 2, & 0 \le t < 1, \\ t, & t \ge 1. \end{cases}$

9. $f(t) = \begin{cases} te^t, & 0 \le t < 1, \\ e^t, & t \ge 1. \end{cases}$

10. $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1, \\ e^{-2t}, & t \ge 1. \end{cases}$

11. $f(t) = \begin{cases} -t, & 0 \le t < 2, \\ t - 4, & 2 \le t < 3, \\ 1, & t \ge 3. \end{cases}$

12. $f(t) = \begin{cases} 0, & 0 \le t < 1, \\ t, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$

13. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ t^2, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$

14. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2, \\ 6, & t > 2. \end{cases}$

15. C/G $f(t) = \begin{cases} \sin t, & 0 \le t < \dfrac{\pi}{2}, \\ 2\sin t, & \dfrac{\pi}{2} \le t < \pi, \\ \cos t, & t \ge \pi. \end{cases}$

16. C/G $f(t) = \begin{cases} 2, & 0 \le t < 1, \\ -2t + 2, & 1 \le t < 3, \\ 3t, & t \ge 3. \end{cases}$

17. C/G $f(t) = \begin{cases} 3, & 0 \le t < 2, \\ 3t + 2, & 2 \le t < 4, \\ 4t, & t \ge 4. \end{cases}$

18. C/G $f(t) = \begin{cases} (t+1)^2, & 0 \le t < 1, \\ (t+2)^2, & t \ge 1. \end{cases}$

In Exercises 19–28 use Theorem 8.4.2 to express the inverse transforms in terms of step functions, and then find distinct formulas the for inverse transforms on the appropriate intervals, as in Example 8.4.7. Where indicated by C/G *, graph the inverse transform.*

19. $H(s) = \dfrac{e^{-2s}}{s - 2}$

20. $H(s) = \dfrac{e^{-s}}{s(s+1)}$

21. C/G $H(s) = \dfrac{e^{-s}}{s^3} + \dfrac{e^{-2s}}{s^2}$

22. C/G $H(s) = \left(\dfrac{2}{s} + \dfrac{1}{s^2}\right) + e^{-s}\left(\dfrac{3}{s} - \dfrac{1}{s^2}\right) + e^{-3s}\left(\dfrac{1}{s} + \dfrac{1}{s^2}\right)$

23. $H(s) = \left(\frac{5}{s} - \frac{1}{s^2}\right) + e^{-3s}\left(\frac{6}{s} + \frac{7}{s^2}\right) + \frac{3e^{-6s}}{s^3}$

24. $H(s) = \frac{e^{-\pi s}(1-2s)}{s^2+4s+5}$

25. C/G $H(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right) + e^{-\frac{\pi}{2}s}\left(\frac{3s-1}{s^2+1}\right)$

26. $H(s) = e^{-2s}\left[\frac{3(s-3)}{(s+1)(s-2)} - \frac{s+1}{(s-1)(s-2)}\right]$

27. $H(s) = \frac{1}{s} + \frac{1}{s^2} + e^{-s}\left(\frac{3}{s} + \frac{2}{s^2}\right) + e^{-3s}\left(\frac{4}{s} + \frac{3}{s^2}\right)$

28. $H(s) = \frac{1}{s} - \frac{2}{s^3} + e^{-2s}\left(\frac{3}{s} - \frac{1}{s^3}\right) + \frac{e^{-4s}}{s^2}$

29. Find $L\left(u(t-\tau)\right)$.

30. Let $\{t_m\}_{m=0}^{\infty}$ be a sequence of points such that $t_0 = 0$, $t_{m+1} > t_m$, and $\lim_{m\to\infty} t_m = \infty$. For each nonnegative integer m, let f_m be continuous on $[t_m, \infty)$, and let f be defined on $[0, \infty)$ by

$$f(t) = f_m(t),\ t_m \le t < t_{m+1} \quad (m = 0, 1, \dots).$$

Show that f is piecewise continuous on $[0, \infty)$ and that it has the step function representation

$$f(t) = f_0(t) + \sum_{m=1}^{\infty} u(t-t_m)\left(f_m(t) - f_{m-1}(t)\right),\ 0 \le t < \infty.$$

How do we know that the series on the right converges for all t in $[0, \infty)$?

31. In addition to the assumptions of Exercise 30, assume that

$$|f_m(t)| \le Me^{s_0 t},\ t \ge t_m,\ m = 0, 1, \dots, \tag{A}$$

and that the series

$$\sum_{m=0}^{\infty} e^{-\rho t_m} \tag{B}$$

converges for some $\rho > 0$. Using the steps listed below, show that $L(f)$ is defined for $s > s_0$ and

$$L(f) = L(f_0) + \sum_{m=1}^{\infty} e^{-st_m} L(g_m) \tag{C}$$

for $s > s_0 + \rho$, where

$$g_m(t) = f_m(t+t_m) - f_{m-1}(t+t_m).$$

(a) Use (A) and Theorem 8.1.6 to show that

$$L(f) = \sum_{m=0}^{\infty} \int_{t_m}^{t_{m+1}} e^{-st} f_m(t)\,dt \tag{D}$$

is defined for $s > s_0$.

(b) Show that (D) can be rewritten as

$$L(f)=\sum_{m=0}^{\infty}\left(\int_{t_m}^{\infty} e^{-st} f_m(t)\,dt-\int_{t_{m+1}}^{\infty} e^{-st} f_m(t)\,dt\right). \tag{E}$$

(c) Use (A), the assumed convergence of (B), and the comparison test to show that the series

$$\sum_{m=0}^{\infty}\int_{t_m}^{\infty} e^{-st} f_m(t)\,dt \quad \text{and} \quad \sum_{m=0}^{\infty}\int_{t_{m+1}}^{\infty} e^{-st} f_m(t)\,dt$$

both converge (absolutely) if $s > s_0 + \rho$.

(d) Show that (E) can be rewritten as

$$L(f)=L(f_0)+\sum_{m=1}^{\infty}\int_{t_m}^{\infty} e^{-st}\left(f_m(t)-f_{m-1}(t)\right)\,dt$$

if $s > s_0 + \rho$.

(e) Complete the proof of (C).

32. Suppose $\{t_m\}_{m=0}^{\infty}$ and $\{f_m\}_{m=0}^{\infty}$ satisfy the assumptions of Exercises 30 and 31, and there's a positive constant K such that $t_m \ge Km$ for m sufficiently large. Show that the series (B) of Exercise 31 converges for any $\rho > 0$, and conclude from this that (C) of Exercise 31 holds for $s > s_0$.

In Exercises 33–36 find the step function representation of f and use the result of Exercise 32 to find $L(f)$. HINT*: You will need formulas related to the formula for the sum of a geometric series.*

33. $f(t)=m+1,\ m\le t<m+1\ (m=0,1,2,\dots)$

34. $f(t)=(-1)^m,\ m\le t<m+1\ (m=0,1,2,\dots)$

35. $f(t)=(m+1)^2,\ m\le t<m+1\ (m=0,1,2,\dots)$

36. $f(t)=(-1)^m m,\ m\le t<m+1\ (m=0,1,2,\dots)$

8.5 CONSTANT COEEFFICIENT EQUATIONS WITH PIECEWISE CONTINUOUS FORCING FUNCTIONS

We'll now consider initial value problems of the form

$$ay''+by'+cy=f(t), \quad y(0)=k_0, \quad y'(0)=k_1, \tag{8.5.1}$$

where a, b, and c are constants ($a \ne 0$) and f is piecewise continuous on $[0,\infty)$. Problems of this kind occur in situations where the input to a physical system undergoes instantaneous changes, as when a switch is turned on or off or the forces acting on the system change abruptly.

It can be shown (Exercises 23 and 24) that the differential equation in (8.5.1) has no solutions on an open interval that contains a jump discontinuity of f. Therefore we must define what we mean by a solution of (8.5.1) on $[0,\infty)$ in the case where f has jump discontinuities. The next theorem motivates our definition. We omit the proof.

Theorem 8.5.1 *Suppose $a, b,$ and c are constants ($a \neq 0$), and f is piecewise continuous on $[0, \infty)$. with jump discontinuities at $t_1, \dots, t_n$, where*

$$0 < t_1 < \cdots < t_n.$$

Let k_0 and k_1 be arbitrary real numbers. Then there is a unique function y defined on $[0, \infty)$ with these properties:

(a) *$y(0) = k_0$ and $y'(0) = k_1$.*

(b) *y and y' are continuous on $[0, \infty)$.*

(c) *y'' is defined on every open subinterval of $[0, \infty)$ that does not contain any of the points $t_1, \dots, t_n$, and*

$$ay'' + by' + cy = f(t)$$

on every such subinterval.

(d) *y'' has limits from the right and left at $t_1, \dots, t_n$.*

We define the function y of Theorem 8.5.1 to be the solution of the initial value problem (8.5.1).

We begin by considering initial value problems of the form

$$ay'' + by' + cy = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t \ge t_1, \end{cases} \qquad y(0) = k_0, \quad y'(0) = k_1, \tag{8.5.2}$$

where the forcing function has a single jump discontinuity at t_1.

We can solve (8.5.2) by the these steps:

Step 1. Find the solution y_0 of the initial value problem

$$ay'' + by' + cy = f_0(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

Step 2. Compute $c_0 = y_0(t_1)$ and $c_1 = y_0'(t_1)$.

Step 3. Find the solution y_1 of the initial value problem

$$ay'' + by' + cy = f_1(t), \quad y(t_1) = c_0, \quad y'(t_1) = c_1.$$

Step 4. Obtain the solution y of (8.5.2) as

$$y = \begin{cases} y_0(t), & 0 \le t < t_1 \\ y_1(t), & t \ge t_1. \end{cases}$$

It is shown in Exercise 23 that y' exists and is continuous at t_1. The next example illustrates this procedure.

Example 8.5.1 Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2, \; y'(0) = -1, \tag{8.5.3}$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < \dfrac{\pi}{2}, \\ -1, & t \ge \dfrac{\pi}{2}. \end{cases}$$

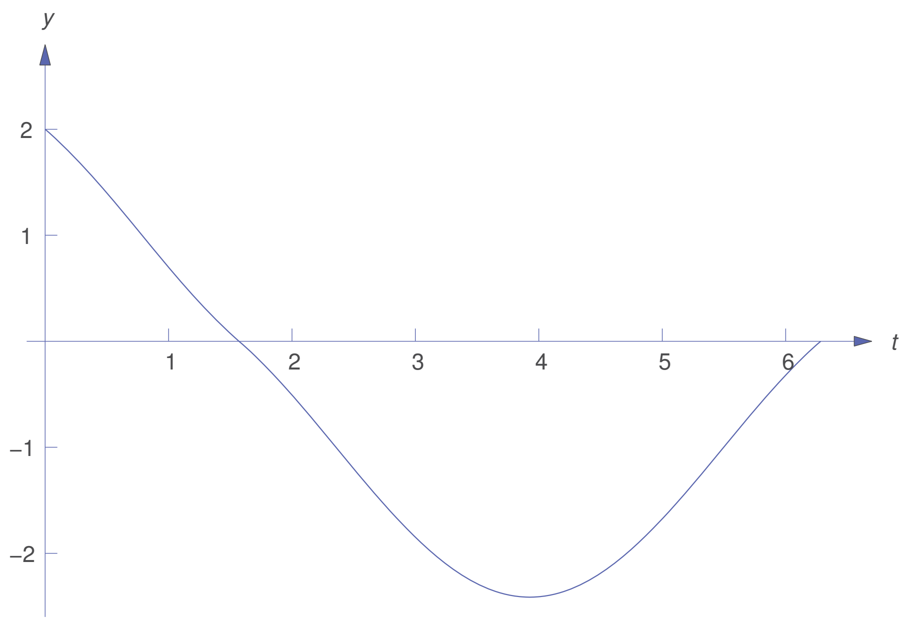

Figure 8.5.1 Graph of (8.5.4)

Solution The initial value problem in Step 1 is

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = -1.$$

We leave it to you to verify that its solution is

$$y_0 = 1 + \cos t - \sin t.$$

Doing Step 2 yields $y_0(\pi/2) = 0$ and $y_0'(\pi/2) = -1$, so the second initial value problem is

$$y'' + y = -1, \quad y\left(\frac{\pi}{2}\right) = 0, \; y'\left(\frac{\pi}{2}\right) = -1.$$

We leave it to you to verify that the solution of this problem is

$$y_1 = -1 + \cos t + \sin t.$$

Hence, the solution of (8.5.3) is

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \dfrac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \ge \dfrac{\pi}{2} \end{cases} \tag{8.5.4}$$

(Figure:8.5.1).

If f_0 and f_1 are defined on $[0, \infty)$, we can rewrite (8.5.2) as

$$ay'' + by' + cy = f_0(t) + u(t - t_1)\left(f_1(t) - f_0(t)\right), \quad y(0) = k_0, \quad y'(0) = k_1,$$

and apply the method of Laplace transforms. We'll now solve the problem considered in Example 8.5.1 by this method.

Example 8.5.2 Use the Laplace transform to solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2,\ y'(0) = -1, \tag{8.5.5}$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < \dfrac{\pi}{2}, \\ -1, & t \ge \dfrac{\pi}{2}. \end{cases}$$

Solution Here

$$f(t) = 1 - 2u\left(t - \frac{\pi}{2}\right),$$

so Theorem 8.4.1 (with $g(t) = 1$) implies that

$$L(f) = \frac{1 - 2e^{-\pi s/2}}{s}.$$

Therefore, transforming (8.5.5) yields

$$(s^2 + 1)Y(s) = \frac{1 - 2e^{-\pi s/2}}{s} - 1 + 2s,$$

so

$$Y(s) = (1 - 2e^{-\pi s/2})G(s) + \frac{2s - 1}{s^2 + 1}, \tag{8.5.6}$$

with

$$G(s) = \frac{1}{s(s^2 + 1)}.$$

The form for the partial fraction expansion of G is

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}. \tag{8.5.7}$$

Multiplying through by $s(s^2 + 1)$ yields

$$A(s^2 + 1) + (Bs + C)s = 1,$$

or

$$(A + B)s^2 + Cs + A = 1.$$

Equating coefficients of like powers of s on the two sides of this equation shows that $A = 1$, $B = -A = -1$ and $C = 0$. Hence, from (8.5.7),

$$G(s) = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Therefore

$$g(t) = 1 - \cos t.$$

From this, (8.5.6), and Theorem 8.4.2,

$$y = 1 - \cos t - 2u\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right) + 2\cos t - \sin t.$$

Simplifying this (recalling that $\cos(t - \pi/2) = \sin t$) yields

$$y = 1 + \cos t - \sin t - 2u\left(t - \frac{\pi}{2}\right)(1 - \sin t),$$

or

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \dfrac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \ge \dfrac{\pi}{2}, \end{cases}$$

which is the result obtained in Example 8.5.1.

REMARK: It isn't obvious that using the Laplace transform to solve (8.5.2) as we did in Example 8.5.2 yields a function y with the properties stated in Theorem 8.5.1; that is, such that y and y' are continuous on $[0, \infty)$ and y'' has limits from the right and left at t_1. However, this is true if f_0 and f_1 are continuous and of exponential order on $[0, \infty)$. A proof is sketched in Exercises 8.6.11–8.6.13.

Example 8.5.3 Solve the initial value problem

$$y'' - y = f(t), \quad y(0) = -1, \; y'(0) = 2, \tag{8.5.8}$$

where

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$$

Solution Here

$$f(t) = t - u(t-1)(t-1),$$

so

$$\begin{aligned} L(f) &= L(t) - L\left(u(t-1)(t-1)\right) \\ &= L(t) - e^{-s}L(t) \text{ (from Theorem 8.4.1)} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \end{aligned}$$

Since transforming (8.5.8) yields

$$(s^2-1)Y(s) = L(f) + 2 - s,$$

we see that

$$Y(s) = (1 - e^{-s})H(s) + \frac{2-s}{s^2-1}, \tag{8.5.9}$$

where

$$H(s) = \frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2};$$

therefore

$$h(t) = \sinh t - t. \tag{8.5.10}$$

Since

$$L^{-1}\left(\frac{2-s}{s^2-1}\right) = 2\sinh t - \cosh t,$$

we conclude from (8.5.9), (8.5.10), and Theorem 8.4.1 that

$$y = \sinh t - t - u(t-1)\left(\sinh(t-1) - t + 1\right) + 2\sinh t - \cosh t,$$

or

$$y = 3\sinh t - \cosh t - t - u(t-1)\left(\sinh(t-1) - t + 1\right) \tag{8.5.11}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = 1$.

Example 8.5.4 Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0,\ y'(0) = 0, \tag{8.5.12}$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < \dfrac{\pi}{4}, \\ \cos 2t, & \dfrac{\pi}{4} \le t < \pi, \\ 0, & t \ge \pi. \end{cases}$$

Solution Here

$$f(t) = u(t - \pi/4)\cos 2t - u(t - \pi)\cos 2t,$$

so

$$\begin{aligned} L(f) &= L\left(u(t - \pi/4)\cos 2t\right) - L\left(u(t - \pi)\cos 2t\right) \\ &= e^{-\pi s/4} L\left(\cos 2(t + \pi/4)\right) - e^{-\pi s} L\left(\cos 2(t + \pi)\right) \\ &= -e^{-\pi s/4} L(\sin 2t) - e^{-\pi s} L(\cos 2t) \\ &= -\frac{2e^{-\pi s/4}}{s^2 + 4} - \frac{se^{-\pi s}}{s^2 + 4}. \end{aligned}$$

Since transforming (8.5.12) yields

$$(s^2 + 1)Y(s) = L(f),$$

we see that

$$Y(s) = e^{-\pi s/4} H_1(s) + e^{-\pi s} H_2(s), \tag{8.5.13}$$

where

$$H_1(s) = -\frac{2}{(s^2 + 1)(s^2 + 4)} \quad \text{and} \quad H_2(s) = -\frac{s}{(s^2 + 1)(s^2 + 4)}. \tag{8.5.14}$$

To simplify the required partial fraction expansions, we first write

$$\frac{1}{(x + 1)(x + 4)} = \frac{1}{3}\left[\frac{1}{x + 1} - \frac{1}{x + 4}\right].$$

Setting $x = s^2$ and substituting the result in (8.5.14) yields

$$H_1(s) = -\frac{2}{3}\left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4}\right] \quad \text{and} \quad H_2(s) = -\frac{1}{3}\left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + 4}\right].$$

The inverse transforms are

$$h_1(t) = -\frac{2}{3}\sin t + \frac{1}{3}\sin 2t \quad \text{and} \quad h_2(t) = -\frac{1}{3}\cos t + \frac{1}{3}\cos 2t.$$

From (8.5.13) and Theorem 8.4.2,

$$y = u\left(t - \frac{\pi}{4}\right) h_1\left(t - \frac{\pi}{4}\right) + u(t - \pi) h_2(t - \pi). \tag{8.5.15}$$

Since

$$\begin{aligned} h_1\left(t - \frac{\pi}{4}\right) &= -\frac{2}{3}\sin\left(t - \frac{\pi}{4}\right) + \frac{1}{3}\sin 2\left(t - \frac{\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t \end{aligned}$$

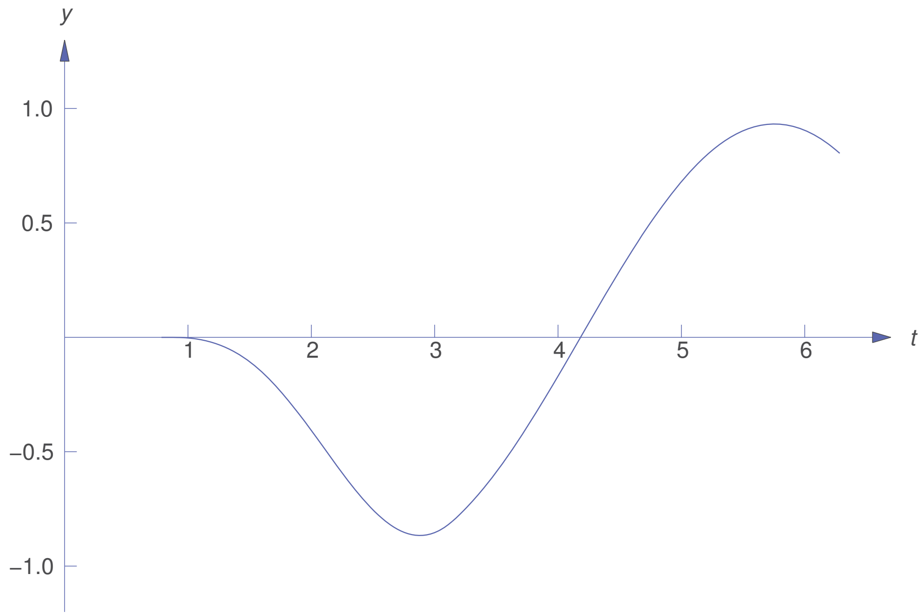

Figure 8.5.2 Graph of (8.5.16)

and

$$\begin{aligned} h_2(t-\pi) &= -\frac{1}{3}\cos(t-\pi)+\frac{1}{3}\cos 2(t-\pi) \\ &= \frac{1}{3}\cos t+\frac{1}{3}\cos 2t, \end{aligned}$$

(8.5.15) can be rewritten as

$$y=-\frac{1}{3}u\left(t-\frac{\pi}{4}\right)\left(\sqrt{2}(\sin t-\cos t)+\cos 2t\right)+\frac{1}{3}u(t-\pi)(\cos t+\cos 2t)$$

or

$$y=\left\{\begin{array}{cl} 0, & 0\le t<\dfrac{\pi}{4}, \\ -\dfrac{\sqrt{2}}{3}(\sin t-\cos t)-\dfrac{1}{3}\cos 2t, & \dfrac{\pi}{4}\le t<\pi, \\ -\dfrac{\sqrt{2}}{3}\sin t+\dfrac{1+\sqrt{2}}{3}\cos t, & t\ge\pi. \end{array}\right. \tag{8.5.16}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1=\pi/4$ and $t_2=\pi$ (Figure 8.5.2).

8.5 Exercises

In Exercises 1–20 use the Laplace transform to solve the initial value problem. Where indicated by C/G , graph the solution.

1. $y'' + y = \begin{cases} 3, & 0 \le t < \pi, \\ 0, & t \ge \pi, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$

2. $y'' + y = \begin{cases} 3, & 0 \le t < 4, \\ ;2t - 5, & t > 4, \end{cases} \quad y(0) = 1, \quad y'(0) = 0$

3. $y'' - 2y' = \begin{cases} 4, & 0 \le t < 1, \\ 6, & t \ge 1, \end{cases} \quad y(0) = -6, \quad y'(0) = 1$

4. $y'' - y = \begin{cases} e^{2t}, & 0 \le t < 2, \\ 1, & t \ge 2, \end{cases} \quad y(0) = 3, \quad y'(0) = -1$

5. $y'' - 3y' + 2y = \begin{cases} 0, & 0 \le t < 1, \\ 1, & 1 \le t < 2, \\ -1, & t \ge 2, \end{cases} \quad y(0) = -3, \quad y'(0) = 1$

6. C/G $y'' + 4y = \begin{cases} |\sin t|, & 0 \le t < 2\pi, \\ 0, & t \ge 2\pi, \end{cases} \quad y(0) = -3, \quad y'(0) = 1$

7. $y'' - 5y' + 4y = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2, \\ 0, & t \ge 2, \end{cases} \quad y(0) = 3, \quad y'(0) = -5$

8. $y'' + 9y = \begin{cases} \cos t, & 0 \le t < \dfrac{3\pi}{2}, \\ \sin t, & t \ge \dfrac{3\pi}{2}, \end{cases} \quad y(0) = 0, \; y'(0) = 0$

9. C/G $y'' + 4y = \begin{cases} t, & 0 \le t < \dfrac{\pi}{2}, \\ \pi, & t \ge \dfrac{\pi}{2}, \end{cases} \quad y(0) = 0, \quad y'(0) = 0$

10. $y'' + y = \begin{cases} t, & 0 \le t < \pi, \\ -t, & t \ge \pi, \end{cases} \quad y(0) = 0, \; y'(0) = 0$

11. $y'' - 3y' + 2y = \begin{cases} 0, & 0 \le t < 2, \\ 2t - 4, & t \ge 2, \end{cases}, \quad y(0) = 0, \quad y'(0) = 0$

12. $y'' + y = \begin{cases} t, & 0 \le t < 2\pi, \\ -2t, & t \ge 2\pi, \end{cases} \quad y(0) = 1, \quad y'(0) = 2$

13. C/G $y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 2, \\ -1, & t \ge 2, \end{cases} \quad y(0) = 0, \; y'(0) = 0$

14. $y'' - 4y' + 3y = \begin{cases} -1, & 0 \le t < 1, \\ 1, & t \ge 1, \end{cases} \quad y(0) = 0, \; y'(0) = 0$

15. $y'' + 2y' + y = \begin{cases} e^t, & 0 \le t < 1, \\ e^t - 1, & t \ge 1, \end{cases} \quad y(0) = 3, \; y'(0) = -1$

16. $y'' + 2y' + y = \begin{cases} 4e^t, & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases} \quad y(0) = 0, \; y'(0) = 0$

17. $y'' + 3y' + 2y = \begin{cases} e^{-t}, & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases} \quad y(0) = 1, \; y'(0) = -1$

18. $y'' - 4y' + 4y = \begin{cases} e^{2t}, & 0 \le t < 2, \\ -e^{2t}, & t \ge 2, \end{cases} \quad y(0) = 0, \; y'(0) = -1$

19. C/G $y'' = \begin{cases} t^2, & 0 \le t < 1, \\ -t, & 1 \le t < 2, \\ t+1, & t \ge 2, \end{cases} \quad y(0) = 1, \; y'(0) = 0$

20. $y'' + 2y' + 2y = \begin{cases} 1, & 0 \le t < 2\pi, \\ t, & 2\pi \le t < 3\pi, \\ -1, & t \ge 3\pi, \end{cases} \quad y(0) = 2, \quad y'(0) = -1$

21. Solve the initial value problem

$$y'' = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = m + 1, \quad m \le t < m + 1, \quad m = 0, 1, 2, \dots.$$

22. Solve the given initial value problem and find a formula that does not involve step functions and represents y on each interval of continuity of f.

(a) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m + 1, \quad m\pi \le t < (m + 1)\pi, \quad m = 0, 1, 2, \dots.$

(b) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (m + 1)t, \quad 2m\pi \le t < 2(m + 1)\pi, \quad m = 0, 1, 2, \dots$ HINT: *You'll need the formula*

$$1 + 2 + \cdots + m = \frac{m(m + 1)}{2}.$$

(c) $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (-1)^m, \quad m\pi \le t < (m + 1)\pi, \quad m = 0, 1, 2, \dots.$

(d) $y'' - y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m + 1, \quad m \le t < (m + 1), \quad m = 0, 1, 2, \dots.$
HINT: *You will need the formula*

$$1 + r + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r} \quad (r \ne 1).$$

(e) $y'' + 2y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (m + 1)(\sin t + 2\cos t), \quad 2m\pi \le t < 2(m + 1)\pi, \quad m = 0, 1, 2, \dots.$
(See the hint in **(d)**.)

(f) $y'' - 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m + 1, \quad m \le t < m + 1, \quad m = 0, 1, 2, \dots.$
(See the hints in **(b)** and **(d)**.)

23. **(a)** Let g be continuous on (α, β) and differentiable on the (α, t_0) and (t_0, β). Suppose $A = \lim_{t \to t_0-} g'(t)$ and $B = \lim_{t \to t_0+} g'(t)$ both exist. Use the mean value theorem to show that

$$\lim_{t \to t_0-} \frac{g(t) - g(t_0)}{t - t_0} = A \quad \text{and} \quad \lim_{t \to t_0+} \frac{g(t) - g(t_0)}{t - t_0} = B.$$

(b) Conclude from **(a)** that $g'(t_0)$ exists and g' is continuous at t_0 if $A = B$.

(c) Conclude from **(a)** that if g is differentiable on (α, β) then g' can't have a jump discontinuity on (α, β).

24. **(a)** Let a, b, and c be constants, with $a \neq 0$. Let f be piecewise continuous on an interval (α, β), with a single jump discontinuity at a point t_0 in (α, β). Suppose y and y' are continuous on (α, β) and y'' on (α, t_0) and (t_0, β). Suppose also that

$$ay'' + by' + cy = f(t) \tag{A}$$

on (α, t_0) and (t_0, β). Show that

$$y''(t_0+) - y''(t_0-) = \frac{f(t_0+) - f(t_0-)}{a} \neq 0.$$

(b) Use **(a)** and Exercise 23**(c)** to show that (A) does not have solutions on any interval (α, β) that contains a jump discontinuity of f.

25. Suppose P_0, P_1, and P_2 are continuous and P_0 has no zeros on an open interval (a, b), and that F has a jump discontinuity at a point t_0 in (a, b). Show that the differential equation

$$P_0(t)y'' + P_1(t)y' + P_2(t)y = F(t)$$

has no solutions on (a, b).HINT: *Generalize the result of Exercise 24 and use Exercise 23*(**c**).

26. Let $0 = t_0 < t_1 < \cdots < t_n$. Suppose f_m is continuous on $[t_m, \infty)$ for $m = 1, \dots, n$. Let

$$f(t) = \begin{cases} f_m(t), & t_m \le t < t_{m+1}, \quad m = 1, \dots, n-1, \\ f_n(t), & t \ge t_n. \end{cases}$$

Show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

as defined following Theorem 8.5.1, is given by

$$y = \begin{cases} z_0(t), & 0 \le t < t_1, \\ z_0(t) + z_1(t), & t_1 \le t < t_2, \\ \vdots & \\ z_0 + \cdots + z_{n-1}(t), & t_{n-1} \le t < t_n, \\ z_0 + \cdots + z_n(t), & t \ge t_n, \end{cases}$$

where z_0 is the solution of

$$az'' + bz' + cz = f_0(t), \quad z(0) = k_0, \quad z'(0) = k_1$$

and z_m is the solution of

$$az'' + bz' + cz = f_m(t) - f_{m-1}(t), \quad z(t_m) = 0, \quad z'(t_m) = 0$$

for $m = 1, \dots, n$.

8.6 CONVOLUTION

In this section we consider the problem of finding the inverse Laplace transform of a product $H(s) = F(s)G(s)$, where F and G are the Laplace transforms of known functions f and g. To motivate our interest in this problem, consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Taking Laplace transforms yields

$$(as^2 + bs + c)Y(s) = F(s),$$

so

$$Y(s) = F(s)G(s), \tag{8.6.1}$$

where

$$G(s) = \frac{1}{as^2 + bs + c}.$$

Until now wen't been interested in the factorization indicated in (8.6.1), since we dealt only with differential equations with specific forcing functions. Hence, we could simply do the indicated multiplication in (8.6.1) and use the table of Laplace transforms to find $y = L^{-1}(Y)$. However, this isn't possible if we want a *formula* for y in terms of f, which may be unspecified.

To motivate the formula for $L^{-1}(FG)$, consider the initial value problem

$$y' - ay = f(t), \quad y(0) = 0, \tag{8.6.2}$$

which we first solve without using the Laplace transform. The solution of the differential equation in (8.6.2) is of the form $y = ue^{at}$ where

$$u' = e^{-at} f(t).$$

Integrating this from 0 to t and imposing the initial condition $u(0) = y(0) = 0$ yields

$$u = \int_0^t e^{-a\tau} f(\tau)\, d\tau.$$

Therefore

$$y(t) = e^{at} \int_0^t e^{-a\tau} f(\tau)\, d\tau = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau. \tag{8.6.3}$$

Now we'll use the Laplace transform to solve (8.6.2) and compare the result to (8.6.3). Taking Laplace transforms in (8.6.2) yields

$$(s - a)Y(s) = F(s),$$

so

$$Y(s) = F(s)\frac{1}{s - a},$$

which implies that

$$y(t) = L^{-1}\left(F(s)\frac{1}{s - a}\right). \tag{8.6.4}$$

If we now let $g(t) = e^{at}$, so that

$$G(s) = \frac{1}{s - a},$$

then (8.6.3) and (8.6.4) can be written as

$$y(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau$$

and

$$y = L^{-1}(FG),$$

respectively. Therefore

$$L^{-1}(FG) = \int_0^t f(\tau)g(t-\tau)\,d\tau \tag{8.6.5}$$

in this case.

This motivates the next definition.

Definition 8.6.1 The *convolution* $f * g$ of two functions f and g is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

It can be shown (Exercise 6) that $f * g = g * f$; that is,

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Eqn. (8.6.5) shows that $L^{-1}(FG) = f * g$ in the special case where $g(t) = e^{at}$. This next theorem states that this is true in general.

Theorem 8.6.2 [*The Convolution Theorem*] *If $L(f) = F$ and $L(g) = G$, then*

$$L(f * g) = FG.$$

A complete proof of the convolution theorem is beyond the scope of this book. However, we'll assume that $f * g$ has a Laplace transform and verify the conclusion of the theorem in a purely computational way. By the definition of the Laplace transform,

$$L(f * g) = \int_0^\infty e^{-st}(f * g)(t)\,dt = \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,d\tau\,dt.$$

This iterated integral equals a double integral over the region shown in Figure 8.6.1. Reversing the order of integration yields

$$L(f * g) = \int_0^\infty f(\tau)\int_\tau^\infty e^{-st}g(t-\tau)\,dt\,d\tau. \tag{8.6.6}$$

However, the substitution $x = t - \tau$ shows that

$$\begin{aligned}
\int_\tau^\infty e^{-st}g(t-\tau)\,dt &= \int_0^\infty e^{-s(x+\tau)}g(x)\,dx \\
&= e^{-s\tau}\int_0^\infty e^{-sx}g(x)\,dx = e^{-s\tau}G(s).
\end{aligned}$$

Substituting this into (8.6.6) and noting that $G(s)$ is independent of τ yields

$$\begin{aligned}
L(f * g) &= \int_0^\infty e^{-s\tau}f(\tau)G(s)\,d\tau \\
&= G(s)\int_0^\infty e^{-st}f(\tau)\,d\tau = F(s)G(s).
\end{aligned}$$

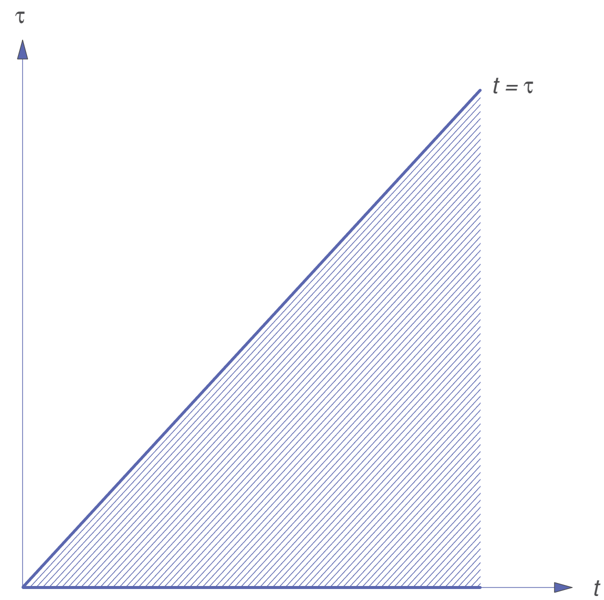

Figure 8.6.1

Example 8.6.1 Let

$$f(t) = e^{at} \quad \text{and} \quad g(t) = e^{bt} \qquad (a \neq b).$$

Verify that $L(f * g) = L(f)L(g)$, as implied by the convolution theorem.

Solution We first compute

$$\begin{aligned}
(f * g)(t) &= \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau &= e^{bt} \int_0^t e^{(a-b)\tau} d\tau \\
&= e^{bt} \left. \frac{e^{(a-b)\tau}}{a-b} \right|_0^t &= \frac{e^{bt}\left[e^{(a-b)t} - 1\right]}{a-b} \\
&= \frac{e^{at} - e^{bt}}{a-b}.
\end{aligned}$$

Since

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad e^{bt} \leftrightarrow \frac{1}{s-b},$$

it follows that

$$\begin{aligned}
L(f * g) &= \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] \\
&= \frac{1}{(s-a)(s-b)} \\
&= L(e^{at})L(e^{bt}) = L(f)L(g).
\end{aligned}$$

A Formula for the Solution of an Initial Value Problem

The convolution theorem provides a formula for the solution of an initial value problem for a linear constant coefficient second order equation with an unspecified. The next three examples illustrate this.

Example 8.6.2 Find a formula for the solution of the initial value problem

$$y'' - 2y' + y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{8.6.7}$$

Solution Taking Laplace transforms in (8.6.7) yields

$$(s^2 - 2s + 1)Y(s) = F(s) + (k_1 + k_0 s) - 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s-1)^2}F(s) + \frac{k_1 + k_0 s - 2k_0}{(s-1)^2} \\ &= \frac{1}{(s-1)^2}F(s) + \frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}. \end{aligned}$$

From the table of Laplace transforms,

$$L^{-1}\left(\frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}\right) = e^t\left(k_0 + (k_1 - k_0)t\right).$$

Since

$$\frac{1}{(s-1)^2} \leftrightarrow te^t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$L^{-1}\left(\frac{1}{(s-1)^2}F(s)\right) = \int_0^t \tau e^\tau f(t-\tau)\,d\tau.$$

Therefore the solution of (8.6.7) is

$$y(t) = e^t\left(k_0 + (k_1 - k_0)t\right) + \int_0^t \tau e^\tau f(t-\tau)\,d\tau.$$

Example 8.6.3 Find a formula for the solution of the initial value problem

$$y'' + 4y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{8.6.8}$$

Solution Taking Laplace transforms in (8.6.8) yields

$$(s^2 + 4)Y(s) = F(s) + k_1 + k_0 s.$$

Therefore

$$Y(s) = \frac{1}{(s^2+4)}F(s) + \frac{k_1 + k_0 s}{s^2 + 4}.$$

From the table of Laplace transforms,

$$L^{-1}\left(\frac{k_1 + k_0 s}{s^2 + 4}\right) = k_0 \cos 2t + \frac{k_1}{2}\sin 2t.$$

Since

$$\frac{1}{(s^2+4)} \leftrightarrow \frac{1}{2}\sin 2t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$L^{-1}\left(\frac{1}{(s^2+4)}F(s)\right) = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\, d\tau.$$

Therefore the solution of (8.6.8) is

$$y(t) = k_0\cos 2t + \frac{k_1}{2}\sin 2t + \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\, d\tau.$$

Example 8.6.4 Find a formula for the solution of the initial value problem

$$y'' + 2y' + 2y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{8.6.9}$$

Solution Taking Laplace transforms in (8.6.9) yields

$$(s^2+2s+2)Y(s) = F(s) + k_1 + k_0 s + 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)^2+1}F(s) + \frac{k_1 + k_0 s + 2k_0}{(s+1)^2+1} \\ &= \frac{1}{(s+1)^2+1}F(s) + \frac{(k_1+k_0) + k_0(s+1)}{(s+1)^2+1}. \end{aligned}$$

From the table of Laplace transforms,

$$L^{-1}\left(\frac{(k_1+k_0)+k_0(s+1)}{(s+1)^2+1}\right) = e^{-t}\left((k_1+k_0)\sin t + k_0\cos t\right).$$

Since

$$\frac{1}{(s+1)^2+1} \leftrightarrow e^{-t}\sin t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$L^{-1}\left(\frac{1}{(s+1)^2+1}F(s)\right) = \int_0^t f(t-\tau)e^{-\tau}\sin\tau\, d\tau.$$

Therefore the solution of (8.6.9) is

$$y(t) = e^{-t}\left((k_1+k_0)\sin t + k_0\cos t\right) + \int_0^t f(t-\tau)e^{-\tau}\sin\tau\, d\tau. \tag{8.6.10}$$

Evaluating Convolution Integrals

We'll say that an integral of the form $\int_0^t u(\tau)v(t-\tau)\, d\tau$ is a *convolution integral*. The convolution theorem provides a convenient way to evaluate convolution integrals.

Example 8.6.5 Evaluate the convolution integral

$$h(t) = \int_0^t (t-\tau)^5 \tau^7 \, d\tau.$$

Solution We could evaluate this integral by expanding $(t-\tau)^5$ in powers of τ and then integrating. However, the convolution theorem provides an easier way. The integral is the convolution of $f(t) = t^5$ and $g(t) = t^7$. Since

$$t^5 \leftrightarrow \frac{5!}{s^6} \quad \text{and} \quad t^7 \leftrightarrow \frac{7!}{s^8},$$

the convolution theorem implies that

$$h(t) \leftrightarrow \frac{5!7!}{s^{14}} = \frac{5!7!}{13!} \frac{13!}{s^{14}},$$

where we have written the second equality because

$$\frac{13!}{s^{14}} \leftrightarrow t^{13}.$$

Hence,

$$h(t) = \frac{5!7!}{13!} t^{13}.$$

Example 8.6.6 Use the convolution theorem and a partial fraction expansion to evaluate the convolution integral

$$h(t) = \int_0^t \sin a(t-\tau) \cos b\tau \, d\tau \quad (|a| \neq |b|).$$

Solution Since

$$\sin at \leftrightarrow \frac{a}{s^2+a^2} \quad \text{and} \quad \cos bt \leftrightarrow \frac{s}{s^2+b^2},$$

the convolution theorem implies that

$$H(s) = \frac{a}{s^2+a^2} \frac{s}{s^2+b^2}.$$

Expanding this in a partial fraction expansion yields

$$H(s) = \frac{a}{b^2-a^2} \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right].$$

Therefore

$$h(t) = \frac{a}{b^2-a^2} \left(\cos at - \cos bt \right).$$

Volterra Integral Equations

An equation of the form

$$y(t) = f(t) + \int_0^t k(t-\tau) y(\tau) \, d\tau \tag{8.6.11}$$

is a *Volterra integral equation*. Here f and k are given functions and y is unknown. Since the integral on the right is a convolution integral, the convolution theorem provides a convenient formula for solving (8.6.11). Taking Laplace transforms in (8.6.11) yields

$$Y(s) = F(s) + K(s)Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{F(s)}{1 - K(s)}.$$

We then obtain the solution of (8.6.11) as $y = L^{-1}(Y)$.

Example 8.6.7 Solve the integral equation

$$y(t) = 1 + 2\int_0^t e^{-2(t-\tau)} y(\tau)\, d\tau. \tag{8.6.12}$$

Solution Taking Laplace transforms in (8.6.12) yields

$$Y(s) = \frac{1}{s} + \frac{2}{s+2} Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{1}{s} + \frac{2}{s^2}.$$

Hence,

$$y(t) = 1 + 2t.$$

Transfer Functions

The next theorem presents a formula for the solution of the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where we assume for simplicity that f is continuous on $[0, \infty)$ and that $L(f)$ exists. In Exercises 11–14 it's shown that the formula is valid under much weaker conditions on f.

Theorem 8.6.3 *Suppose f is continuous on $[0, \infty)$ and has a Laplace transform. Then the solution of the initial value problem*

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \tag{8.6.13}$$

is

$$y(t) = k_0 y_1(t) + k_1 y_2(t) + \int_0^t w(\tau) f(t-\tau)\, d\tau, \tag{8.6.14}$$

where y_1 and y_2 satisfy

$$ay_1'' + by_1' + cy_1 = 0, \quad y_1(0) = 1, \quad y_1'(0) = 0, \tag{8.6.15}$$

and

$$ay_2'' + by_2' + cy_2 = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1, \tag{8.6.16}$$

and

$$w(t) = \frac{1}{a} y_2(t). \tag{8.6.17}$$

Proof Taking Laplace transforms in (8.6.13) yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0,$$

where

$$p(s) = as^2 + bs + c.$$

Hence,

$$Y(s) = W(s)F(s) + V(s) \tag{8.6.18}$$

with

$$W(s) = \frac{1}{p(s)} \tag{8.6.19}$$

and

$$V(s) = \frac{a(k_1 + k_0 s) + bk_0}{p(s)}. \tag{8.6.20}$$

Taking Laplace transforms in (8.6.15) and (8.6.16) shows that

$$p(s)Y_1(s) = as + b \quad \text{and} \quad p(s)Y_2(s) = a.$$

Therefore

$$Y_1(s) = \frac{as + b}{p(s)}$$

and

$$Y_2(s) = \frac{a}{p(s)}. \tag{8.6.21}$$

Hence, (8.6.20) can be rewritten as

$$V(s) = k_0 Y_1(s) + k_1 Y_2(s).$$

Substituting this into (8.6.18) yields

$$Y(s) = k_0 Y_1(s) + k_1 Y_2(s) + \frac{1}{a} Y_2(s)F(s).$$

Taking inverse transforms and invoking the convolution theorem yields (8.6.14). Finally, (8.6.19) and (8.6.21) imply (8.6.17).

It is useful to note from (8.6.14) that y is of the form

$$y = v + h,$$

where

$$v(t) = k_0 y_1(t) + k_1 y_2(t)$$

depends on the initial conditions and is independent of the forcing function, while

$$h(t) = \int_0^t w(\tau) f(t - \tau)\, d\tau$$

depends on the forcing function and is independent of the initial conditions. If the zeros of the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation have negative real parts, then y_1 and y_2 both approach zero as $t \to \infty$, so $\lim_{t\to\infty} v(t) = 0$ for any choice of initial conditions. Moreover, the value of $h(t)$ is essentially independent of the values of $f(t-\tau)$ for large τ, since $\lim_{\tau\to\infty} w(\tau) = 0$. In this case we say that v and h are *transient* and *steady state components*, respectively, of the solution y of (8.6.13). These definitions apply to the initial value problem of Example 8.6.4, where the zeros of

$$p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$$

are $-1 \pm i$. From (8.6.10), we see that the solution of the general initial value problem of Example 8.6.4 is $y = v + h$, where

$$v(t) = e^{-t}\left((k_1 + k_0)\sin t + k_0 \cos t\right)$$

is the transient component of the solution and

$$h(t) = \int_0^t f(t-\tau)e^{-\tau}\sin\tau \, d\tau$$

is the steady state component. The definitions don't apply to the initial value problems considered in Examples 8.6.2 and 8.6.3, since the zeros of the characteristic polynomials in these two examples don't have negative real parts.

In physical applications where the input f and the output y of a device are related by (8.6.13), the zeros of the characteristic polynomial usually do have negative real parts. Then $W = L(w)$ is called the *transfer function* of the device. Since

$$H(s) = W(s)F(s),$$

we see that

$$W(s) = \frac{H(s)}{F(s)}$$

is the ratio of the transform of the steady state output to the transform of the input.

Because of the form of

$$h(t) = \int_0^t w(\tau) f(t-\tau)\, d\tau,$$

w is sometimes called the *weighting function* of the device, since it assigns weights to past values of the input f. It is also called the *impulse response* of the device, for reasons discussed in the next section.

Formula (8.6.14) is given in more detail in Exercises 8–10 for the three possible cases where the zeros of $p(s)$ are real and distinct, real and repeated, or complex conjugates, respectively.

8.6 Exercises

1. Express the inverse transform as an integral.

(a) $\dfrac{1}{s^2(s^2+4)}$ **(b)** $\dfrac{s}{(s+2)(s^2+9)}$

(c) $\dfrac{s}{(s^2+4)(s^2+9)}$ **(d)** $\dfrac{s}{(s^2+1)^2}$

(e) $\dfrac{1}{s(s-a)}$ **(f)** $\dfrac{1}{(s+1)(s^2+2s+2)}$

(g) $\dfrac{1}{(s+1)^2(s^2+4s+5)}$ **(h)** $\dfrac{1}{(s-1)^3(s+2)^2}$

(i) $\dfrac{s-1}{s^2(s^2-2s+2)}$ **(j)** $\dfrac{s(s+3)}{(s^2+4)(s^2+6s+10)}$

(k) $\dfrac{1}{(s-3)^5 s^6}$ **(l)** $\dfrac{1}{(s-1)^3(s^2+4)}$

(m) $\dfrac{1}{s^2(s-2)^3}$ **(n)** $\dfrac{1}{s^7(s-2)^6}$

2. Find the Laplace transform.

(a) $\int_0^t \sin a\tau \cos b(t-\tau)\,d\tau$ **(b)** $\int_0^t e^{\tau} \sin a(t-\tau)\,d\tau$

(c) $\int_0^t \sinh a\tau \cosh a(t-\tau)\,d\tau$ **(d)** $\int_0^t \tau(t-\tau)\sin\omega\tau\cos\omega(t-\tau)\,d\tau$

(e) $e^t\int_0^t \sin\omega\tau\cos\omega(t-\tau)\,d\tau$ **(f)** $e^t\int_0^t \tau^2(t-\tau)e^{\tau}\,d\tau$

(g) $e^{-t}\int_0^t e^{-\tau}\tau\cos\omega(t-\tau)\,d\tau$ **(h)** $e^t\int_0^t e^{2\tau}\sinh(t-\tau)\,d\tau$

(i) $\int_0^t \tau e^{2\tau}\sin 2(t-\tau)\,d\tau$ **(j)** $\int_0^t (t-\tau)^3 e^{\tau}\,d\tau$

(k) $\int_0^t \tau^6 e^{-(t-\tau)}\sin 3(t-\tau)\,d\tau$ **(l)** $\int_0^t \tau^2(t-\tau)^3\,d\tau$

(m) $\int_0^t (t-\tau)^7 e^{-\tau}\sin 2\tau\,d\tau$ **(n)** $\int_0^t (t-\tau)^4\sin 2\tau\,d\tau$

3. Find a formula for the solution of the initial value problem.

(a) $y''+3y'+y=f(t), \quad y(0)=0, \quad y'(0)=0$
(b) $y''+4y=f(t), \quad y(0)=0, \quad y'(0)=0$
(c) $y''+2y'+y=f(t), \quad y(0)=0, \quad y'(0)=0$
(d) $y''+k^2y=f(t), \quad y(0)=1, \quad y'(0)=-1$
(e) $y''+6y'+9y=f(t), \quad y(0)=0, \quad y'(0)=-2$
(f) $y''-4y=f(t), \quad y(0)=0, \quad y'(0)=3$
(g) $y''-5y'+6y=f(t), \quad y(0)=1, \quad y'(0)=3$
(h) $y''+\omega^2y=f(t), \quad y(0)=k_0, \quad y'(0)=k_1$

4. Solve the integral equation.

(a) $y(t)=t-\int_0^t (t-\tau)y(\tau)\,d\tau$

(b) $y(t)=\sin t-2\int_0^t \cos(t-\tau)y(\tau)\,d\tau$

(c) $y(t)=1+2\int_0^t y(\tau)\cos(t-\tau)\,d\tau$ **(d)** $y(t)=t+\int_0^t y(\tau)e^{-(t-\tau)}\,d\tau$

(e) $y'(t)=t+\int_0^t y(\tau)\cos(t-\tau)\,d\tau, \ y(0)=4$

(f) $y(t) = \cos t - \sin t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

5. Use the convolution theorem to evaluate the integral.

(a) $\int_0^t (t-\tau)^7\tau^8\,d\tau$

(b) $\int_0^t (t-\tau)^{13}\tau^7\,d\tau$

(c) $\int_0^t (t-\tau)^6\tau^7\,d\tau$

(d) $\int_0^t e^{-\tau}\sin(t-\tau)\,d\tau$

(e) $\int_0^t \sin\tau\cos 2(t-\tau)\,d\tau$

6. Show that

$$\int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

by introducing the new variable of integration $x = t - \tau$ in the first integral.

7. Use the convolution theorem to show that if $f(t) \leftrightarrow F(s)$ then

$$\int_0^t f(\tau)\,d\tau \leftrightarrow \frac{F(s)}{s}.$$

8. Show that if $p(s) = as^2 + bs + c$ has distinct real zeros r_1 and r_2 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$\begin{aligned} y(t) &= k_0\frac{r_2e^{r_1t} - r_1e^{r_2t}}{r_2 - r_1} + k_1\frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1} \\ &\quad + \frac{1}{a(r_2 - r_1)}\int_0^t (e^{r_2\tau} - e^{r_1\tau})f(t-\tau)\,d\tau. \end{aligned}$$

9. Show that if $p(s) = as^2 + bs + c$ has a repeated real zero r_1 then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$y(t) = k_0(1 - r_1t)e^{r_1t} + k_1te^{r_1t} + \frac{1}{a}\int_0^t \tau e^{r_1\tau}f(t-\tau)\,d\tau.$$

10. Show that if $p(s) = as^2 + bs + c$ has complex conjugate zeros $\lambda \pm i\omega$ then the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1$$

is

$$\begin{aligned} y(t) &= e^{\lambda t}\left[k_0(\cos\omega t - \frac{\lambda}{\omega}\sin\omega t) + \frac{k_1}{\omega}\sin\omega t\right] \\ &\quad + \frac{1}{a\omega}\int_0^t e^{\lambda t}f(t-\tau)\sin\omega\tau\,d\tau. \end{aligned}$$

11. Let

$$w = L^{-1}\left(\frac{1}{as^2 + bs + c}\right),$$

where a, b, and c are constants and $a \neq 0$.

(a) Show that w is the solution of

$$aw'' + bw' + cw = 0, \quad w(0) = 0, \quad w'(0) = \frac{1}{a}.$$

(b) Let f be continuous on $[0, \infty)$ and define

$$h(t) = \int_0^t w(t - \tau) f(\tau)\, d\tau.$$

Use *Leibniz's rule* for differentiating an integral with respect to a parameter to show that h is the solution of

$$ah'' + bh' + ch = f, \quad h(0) = 0, \quad h'(0) = 0.$$

(c) Show that the function y in Eqn. (8.6.14) is the solution of Eqn. (8.6.13) provided that f is continuous on $[0, \infty)$; thus, it's not necessary to assume that f has a Laplace transform.

12. Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{A}$$

where a, b, and c are constants, $a \neq 0$, and

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1. \end{cases}$$

Assume that f_0 is continuous and of exponential order on $[0, \infty)$ and f_1 is continuous and of exponential order on $[t_1, \infty)$. Let

$$p(s) = as^2 + bs + c.$$

(a) Show that the Laplace transform of the solution of (A) is

$$Y(s) = \frac{F_0(s) + e^{-st_1} G(s)}{p(s)}$$

where $g(t) = f_1(t + t_1) - f_0(t + t_1)$.

(b) Let w be as in Exercise 11. Use Theorem 8.4.2 and the convolution theorem to show that the solution of (A) is

$$y(t) = \int_0^t w(t - \tau) f_0(\tau)\, d\tau + u(t - t_1) \int_0^{t - t_1} w(t - t_1 - \tau) g(\tau)\, d\tau$$

for $t > 0$.

(c) Henceforth, assume only that f_0 is continuous on $[0, \infty)$ and f_1 is continuous on $[t_1, \infty)$. Use Exercise 11 **(a)** and **(b)** to show that

$$y'(t) = \int_0^t w'(t - \tau) f_0(\tau)\, d\tau + u(t - t_1) \int_0^{t - t_1} w'(t - t_1 - \tau) g(\tau)\, d\tau$$

for $t > 0$, and

$$y''(t) = \frac{f(t)}{a} + \int_0^t w''(t-\tau)f_0(\tau)\,d\tau + u(t-t_1)\int_0^{t-t_1} w''(t-t_1-\tau)g(\tau)\,d\tau$$

for $0 < t < t_1$ and $t > t_1$. Also, show y satisfies the differential equation in (A) on$(0, t_1)$ and (t_1, ∞).

(d) Show that y and y' are continuous on $[0, \infty)$.

13. Suppose

$$f(t) = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t_1 \le t < t_2, \\ & \vdots \\ f_{k-1}(t), & t_{k-1} \le t < t_k, \\ f_k(t), & t \ge t_k, \end{cases}$$

where f_m is continuous on $[t_m, \infty)$ for $m = 0, \dots, k$ (let $t_0 = 0$), and define

$$g_m(t) = f_m(t + t_m) - f_{m-1}(t + t_m),\ m = 1, \dots, k.$$

Extend the results of Exercise 12 to show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

is

$$y(t) = \int_0^t w(t-\tau)f_0(\tau)\,d\tau + \sum_{m=1}^{k} u(t-t_m)\int_0^{t-t_m} w(t-t_m-\tau)g_m(\tau)\,d\tau.$$

14. Let $\{t_m\}_{m=0}^{\infty}$ be a sequence of points such that $t_0 = 0$, $t_{m+1} > t_m$, and $\lim_{m\to\infty} t_m = \infty$. For each nonegative integer m let f_m be continuous on $[t_m, \infty)$, and let f be defined on $[0, \infty)$ by

$$f(t) = f_m(t), \quad t_m \le t < t_{m+1} \quad m = 0, 1, 2 \dots.$$

Let

$$g_m(t) = f_m(t + t_m) - f_{m-1}(t + t_m), \quad m = 1, \dots, k.$$

Extend the results of Exercise 13 to show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

is

$$y(t) = \int_0^t w(t-\tau)f_0(\tau)\,d\tau + \sum_{m=1}^{\infty} u(t-t_m)\int_0^{t-t_m} w(t-t_m-\tau)g_m(\tau)\,d\tau.$$

HINT: *See Exercise*30.

8.7 CONSTANT COEFFICIENT EQUATIONS WITH IMPULSES

So far in this chapter, we've considered initial value problems for the constant coefficient equation

$$ay'' + by' + cy = f(t),$$

where f is continuous or piecewise continuous on $[0,\infty)$. In this section we consider initial value problems where f represents a force that's very large for a short time and zero otherwise. We say that such forces are *impulsive*. Impulsive forces occur, for example, when two objects collide. Since it isn't feasible to represent such forces as continuous or piecewise continuous functions, we must construct a different mathematical model to deal with them.

If f is an integrable function and $f(t)=0$ for t outside of the interval $[t_0,t_0+h]$, then $\int_{t_0}^{t_0+h} f(t)\,dt$ is called the *total impulse* of f. We're interested in the idealized situation where h is so small that the total impulse can be assumed to be applied instantaneously at $t=t_0$. We say in this case that f is an *impulse function*. In particular, we denote by $\delta(t-t_0)$ the impulse function with total impulse equal to one, applied at $t=t_0$. (The impulse function $\delta(t)$ obtained by setting $t_0=0$ is the *Dirac δ function*.) It must be understood, however, that $\delta(t-t_0)$ isn't a function in the standard sense, since our "definition" implies that $\delta(t-t_0)=0$ if $t\ne t_0$, while

$$\int_{t_0}^{t_0} \delta(t-t_0)\,dt = 1.$$

From calculus we know that no function can have these properties; nevertheless, there's a branch of mathematics known as the *theory of distributions* where the definition can be made rigorous. Since the theory of distributions is beyond the scope of this book, we'll take an intuitive approach to impulse functions.

Our first task is to define what we mean by the solution of the initial value problem

$$ay''+by'+cy=\delta(t-t_0), \quad y(0)=0, \quad y'(0)=0,$$

where t_0 is a fixed nonnegative number. The next theorem will motivate our definition.

Theorem 8.7.1 *Suppose $t_0\ge 0$. For each positive number h, let y_h be the solution of the initial value problem*

$$ay_h''+by_h'+cy_h=f_h(t), \quad y_h(0)=0, \quad y_h'(0)=0, \tag{8.7.1}$$

where

$$f_h(t)=\begin{cases} 0, & 0\le t<t_0,\\ 1/h, & t_0\le t<t_0+h,\\ 0, & t\ge t_0+h,\end{cases} \tag{8.7.2}$$

so f_h has unit total impulse equal to the area of the shaded rectangle in Figure 8.7.1. Then

$$\lim_{h\to 0+} y_h(t)=u(t-t_0)w(t-t_0), \tag{8.7.3}$$

where

$$w=L^{-1}\left(\frac{1}{as^2+bs+c}\right).$$

Proof Taking Laplace transforms in (8.7.1) yields

$$(as^2+bs+c)Y_h(s)=F_h(s),$$

so

$$Y_h(s)=\frac{F_h(s)}{as^2+bs+c}.$$

The convolution theorem implies that

$$y_h(t)=\int_0^t w(t-\tau)f_h(\tau)\,d\tau.$$

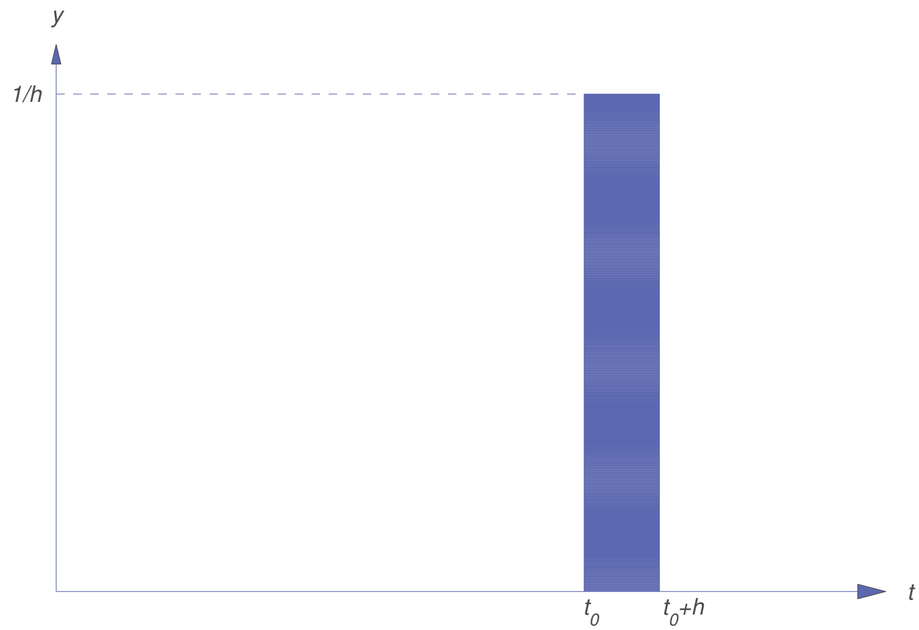

Figure 8.7.1 $y = f_h(t)$

Therefore, (8.7.2) implies that

$$y_h(t) = \begin{cases} 0, & 0 \le t < t_0, \\ \dfrac{1}{h}\displaystyle\int_{t_0}^{t} w(t-\tau)\,d\tau, & t_0 \le t \le t_0 + h, \\ \dfrac{1}{h}\displaystyle\int_{t_0}^{t_0+h} w(t-\tau)\,d\tau, & t > t_0 + h. \end{cases} \tag{8.7.4}$$

Since $y_h(t) = 0$ for all h if $0 \le t \le t_0$, it follows that

$$\lim_{h \to 0+} y_h(t) = 0 \quad \text{if} \quad 0 \le t \le t_0. \tag{8.7.5}$$

We'll now show that

$$\lim_{h \to 0+} y_h(t) = w(t - t_0) \quad \text{if} \quad t > t_0. \tag{8.7.6}$$

Suppose t is fixed and $t > t_0$. From (8.7.4),

$$y_h(t) = \frac{1}{h}\int_{t_0}^{t_0+h} w(t-\tau)d\tau \quad \text{if} \quad h < t - t_0. \tag{8.7.7}$$

Since

$$\frac{1}{h}\int_{t_0}^{t_0+h} d\tau = 1, \tag{8.7.8}$$

we can write

$$w(t-t_0) = \frac{1}{h} w(t-t_0)\int_{t_0}^{t_0+h} d\tau = \frac{1}{h}\int_{t_0}^{t_0+h} w(t-t_0)\,d\tau.$$

From this and (8.7.7),

$$y_h(t) - w(t - t_0) = \frac{1}{h} \int_{t_0}^{t_0+h} (w(t - \tau) - w(t - t_0))\, d\tau.$$

Therefore

$$|y_h(t) - w(t - t_0)| \le \frac{1}{h} \int_{t_0}^{t_0+h} |w(t - \tau) - w(t - t_0)|\, d\tau. \tag{8.7.9}$$

Now let M_h be the maximum value of $|w(t - \tau) - w(t - t_0)|$ as τ varies over the interval $[t_0, t_0 + h]$. (Remember that t and t_0 are fixed.) Then (8.7.8) and (8.7.9) imply that

$$|y_h(t) - w(t - t_0)| \le \frac{1}{h} M_h \int_{t_0}^{t_0+h} d\tau = M_h. \tag{8.7.10}$$

But $\lim_{h\to 0+} M_h = 0$, since w is continuous. Therefore (8.7.10) implies (8.7.6). This and (8.7.5) imply (8.7.3).

Theorem 8.7.1 motivates the next definition.

Definition 8.7.2 If $t_0 > 0$, then the solution of the initial value problem

$$ay'' + by' + cy = \delta(t - t_0), \quad y(0) = 0, \quad y'(0) = 0, \tag{8.7.11}$$

is defined to be

$$y = u(t - t_0)w(t - t_0),$$

where

$$w = \mathcal{L}^{-1}\left(\frac{1}{as^2 + bs + c}\right).$$

In physical applications where the input f and the output y of a device are related by the differential equation

$$ay'' + by' + cy = f(t),$$

w is called the *impulse response* of the device. Note that w is the solution of the initial value problem

$$aw'' + bw' + cw = 0, \quad w(0) = 0, \quad w'(0) = 1/a, \tag{8.7.12}$$

as can be seen by using the Laplace transform to solve this problem. (Verify.) On the other hand, we can solve (8.7.12) by the methods of Section 5.2 and show that w is defined on $(-\infty, \infty)$ by

$$w = \frac{e^{r_2 t} - e^{r_1 t}}{a(r_2 - r_1)}, \quad w = \frac{1}{a} t e^{r_1 t}, \quad \text{or} \quad w = \frac{1}{a\omega} e^{\lambda t} \sin \omega t, \tag{8.7.13}$$

depending upon whether the polynomial $p(r) = ar^2 + br + c$ has distinct real zeros r_1 and r_2, a repeated zero r_1, or complex conjugate zeros $\lambda \pm i\omega$. (In most physical applications, the zeros of the characteristic polynomial have negative real parts, so $\lim_{t\to\infty} w(t) = 0$.) This means that $y = u(t - t_0)w(t - t_0)$ is defined on $(-\infty, \infty)$ and has the following properties:

$$y(t) = 0, \quad t < t_0,$$

$$ay'' + by' + cy = 0 \quad \text{on} \quad (-\infty, t_0) \quad \text{and} \quad (t_0, \infty),$$

and

$$y'_-(t_0) = 0, \quad y'_+(t_0) = 1/a \tag{8.7.14}$$

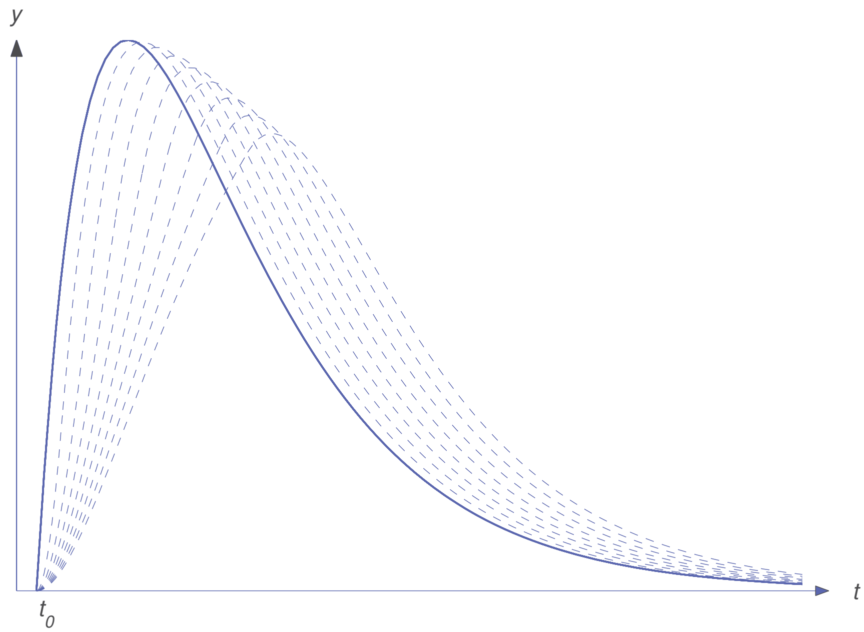

Figure 8.7.2 An illustration of Theorem 8.7.1

(remember that $y'_-(t_0)$ and $y'_+(t_0)$ are derivatives from the right and left, respectively) and $y'(t_0)$ does not exist. Thus, even though we defined $y = u(t-t_0)w(t-t_0)$ to be the solution of (8.7.11), this function *doesn't satisfy* the differential equation in (8.7.11) *at* t_0, since it isn't differentiable there; in fact (8.7.14) indicates that an impulse causes a jump discontinuity in velocity. (To see that this is reasonable, think of what happens when you hit a ball with a bat.) This means that the initial value problem (8.7.11) doesn't make sense if $t_0 = 0$, since $y'(0)$ doesn't exist in this case. However $y = u(t)w(t)$ can be defined to be the solution of the modified initial value problem

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'_-(0) = 0,$$

where the condition on the derivative at $t = 0$ has been replaced by a condition on the derivative from the left.

Figure 8.7.2 illustrates Theorem 8.7.1 for the case where the impulse response w is the first expression in (8.7.13) and r_1 and r_2 are distinct and both negative. The solid curve in the figure is the graph of w. The dashed curves are solutions of (8.7.1) for various values of h. As h decreases the graph of y_h moves to the left toward the graph of w.

Example 8.7.1 Find the solution of the initial value problem

$$y'' - 2y' + y = \delta(t - t_0), \quad y(0) = 0, \quad y'(0) = 0, \tag{8.7.15}$$

where $t_0 > 0$. Then interpret the solution for the case where $t_0 = 0$.

Solution Here

$$w = L^{-1}\left(\frac{1}{s^2 - 2s + 1}\right) = L^{-1}\left(\frac{1}{(s-1)^2}\right) = te^{-t},$$

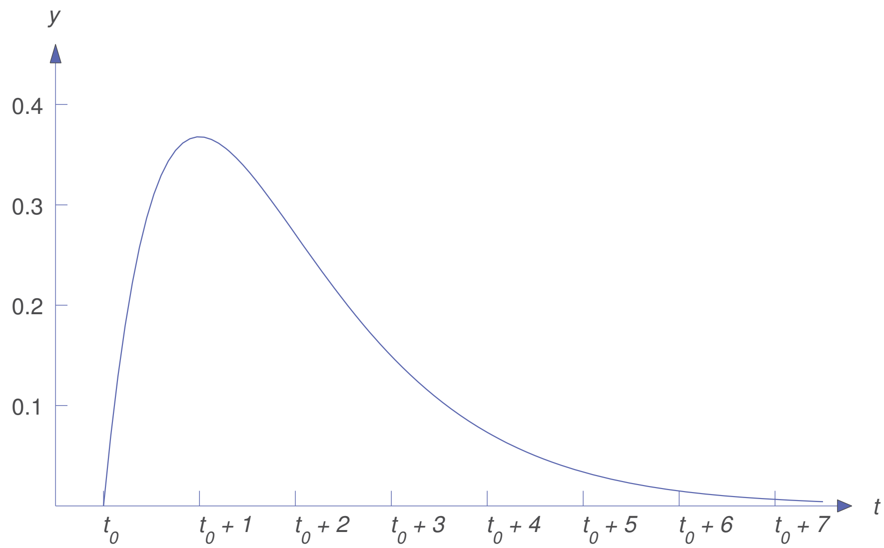

Figure 8.7.3 $y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$

so Definition 8.7.2 yields

$$y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$$

as the solution of (8.7.15) if $t_0 > 0$. If $t_0 = 0$, then (8.7.15) doesn't have a solution; however, $y = u(t)te^{-t}$ (which we would usually write simply as $y = te^{-t}$) is the solution of the modified initial value problem

$$y'' - 2y' + y = \delta(t), \quad y(0) = 0, \quad y'_-(0) = 0.$$

The graph of $y = u(t - t_0)(t - t_0)e^{-(t-t_0)}$ is shown in Figure 8.7.3

Definition 8.7.2 and the principle of superposition motivate the next definition.

Definition 8.7.3 Suppose α is a nonzero constant and f is piecewise continuous on $[0, \infty)$. If $t_0 > 0$, then the solution of the initial value problem

$$ay'' + by' + cy = f(t) + \alpha\delta(t - t_0), \quad y(0) = k_0, \quad y'(0) = k_1$$

is defined to be

$$y(t) = \hat{y}(t) + \alpha u(t - t_0)w(t - t_0),$$

where $\hat{y}$ is the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

This definition also applies if $t_0 = 0$, provided that the initial condition $y'(0) = k_1$ is replaced by $y'_-(0) = k_1$.

Example 8.7.2 Solve the initial value problem

$$y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t - 1), \quad y(0) = -3, \quad y'(0) = 2. \tag{8.7.16}$$

Solution We leave it to you to show that the solution of

$$y'' + 6y' + 5y = 3e^{-2t}, \quad y(0) = -3,\ y'(0) = 2$$

is

$$\hat{y} = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t}.$$

Since

$$\begin{aligned} w(t) &= L^{-1}\left(\frac{1}{s^2+6s+5}\right) &= L^{-1}\left(\frac{1}{(s+1)(s+5)}\right) \\ &= \frac{1}{4}L^{-1}\left(\frac{1}{s+1} - \frac{1}{s+5}\right) &= \frac{e^{-t}-e^{-5t}}{4}, \end{aligned}$$

the solution of (8.7.16) is

$$y = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t} + u(t-1)\frac{e^{-(t-1)} - e^{-5(t-1)}}{2} \tag{8.7.17}$$

(Figure 8.7.4) .

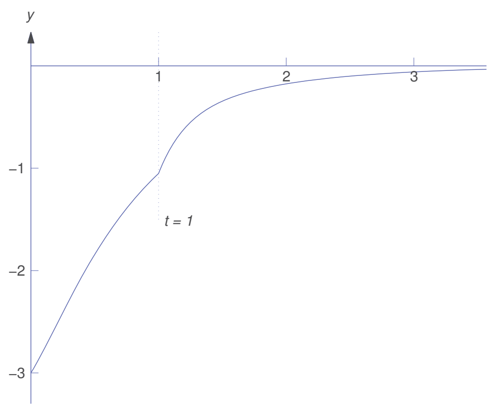

Figure 8.7.4 Graph of (8.7.17)

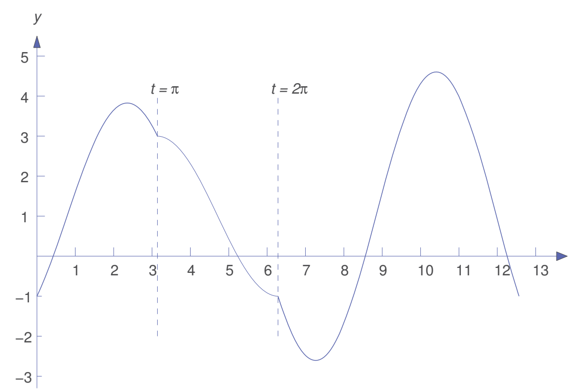

Figure 8.7.5 Graph of (8.7.19)

Definition 8.7.3 can be extended in the obvious way to cover the case where the forcing function contains more than one impulse.

Example 8.7.3 Solve the initial value problem

$$y'' + y = 1 + 2\delta(t-\pi) - 3\delta(t-2\pi), \quad y(0) = -1,\ y'(0) = 2. \tag{8.7.18}$$

Solution We leave it to you to show that

$$\hat{y} = 1 - 2\cos t + 2\sin t$$

is the solution of

$$y'' + y = 1, \quad y(0) = -1, \quad y'(0) = 2.$$

Since

$$w = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t,$$

the solution of (8.7.18) is

$$\begin{aligned} y &= 1 - 2\cos t + 2\sin t + 2u(t-\pi)\sin(t-\pi) - 3u(t-2\pi)\sin(t-2\pi) \\ &= 1 - 2\cos t + 2\sin t - 2u(t-\pi)\sin t - 3u(t-2\pi)\sin t, \end{aligned}$$

or

$$y = \left\{ \begin{array}{cl} 1 - 2\cos t + 2\sin t, & 0 \le t < \pi, \\ 1 - 2\cos t, & \pi \le t < 2\pi, \\ 1 - 2\cos t - 3\sin t, & t \ge 2\pi \end{array} \right. \tag{8.7.19}$$

(Figure 8.7.5).

8.7 Exercises

In Exercises 1–20 solve the initial value problem. Where indicated by C/G, graph the solution.

1. $y'' + 3y' + 2y = 6e^{2t} + 2\delta(t-1), \quad y(0) = 2, \quad y'(0) = -6$

2. C/G $y'' + y' - 2y = -10e^{-t} + 5\delta(t-1), \quad y(0) = 7, \quad y'(0) = -9$

3. $y'' - 4y = 2e^{-t} + 5\delta(t-1), \quad y(0) = -1, \quad y'(0) = 2$

4. C/G $y'' + y = \sin 3t + 2\delta(t-\pi/2), \quad y(0) = 1, \quad y'(0) = -1$

5. $y'' + 4y = 4 + \delta(t-3\pi), \quad y(0) = 0, \quad y'(0) = 1$

6. $y'' - y = 8 + 2\delta(t-2), \quad y(0) = -1, \quad y'(0) = 1$

7. $y'' + y' = e^t + 3\delta(t-6), \quad y(0) = -1, \quad y'(0) = 4$

8. $y'' + 4y = 8e^{2t} + \delta(t-\pi/2), \quad y(0) = 8, \quad y'(0) = 0$

9. C/G $y'' + 3y' + 2y = 1 + \delta(t-1), \quad y(0) = 1, \quad y'(0) = -1$

10. $y'' + 2y' + y = e^t + 2\delta(t-2), \quad y(0) = -1, \quad y'(0) = 2$

11. C/G $y'' + 4y = \sin t + \delta(t-\pi/2), \quad y(0) = 0, \quad y'(0) = 2$

12. $y'' + 2y' + 2y = \delta(t-\pi) - 3\delta(t-2\pi), \quad y(0) = -1, \quad y'(0) = 2$

13. $y'' + 4y' + 13y = \delta(t-\pi/6) + 2\delta(t-\pi/3), \quad y(0) = 1, \quad y'(0) = 2$

14. $2y'' - 3y' - 2y = 1 + \delta(t-2), \quad y(0) = -1, \quad y'(0) = 2$

15. $4y'' - 4y' + 5y = 4\sin t - 4\cos t + \delta(t-\pi/2) - \delta(t-\pi), \quad y(0) = 1, \quad y'(0) = 1$

16. $y'' + y = \cos 2t + 2\delta(t-\pi/2) - 3\delta(t-\pi), \quad y(0) = 0, \quad y'(0) = -1$

17. C/G $y'' - y = 4e^{-t} - 5\delta(t-1) + 3\delta(t-2), \quad y(0) = 0, \quad y'(0) = 0$

18. $y'' + 2y' + y = e^t - \delta(t-1) + 2\delta(t-2), \quad y(0) = 0, \quad y'(0) = -1$

19. $y'' + y = f(t) + \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 1$, and

$$f(t) = \begin{cases} \sin 2t, & 0 \le t < \pi, \\ 0, & t \ge \pi. \end{cases}$$

20. $y'' + 4y = f(t) + \delta(t-\pi) - 3\delta(t-3\pi/2), \quad y(0) = 1, \quad y'(0) = -1$, and

$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2, \\ 2, & t \ge \pi/2 \end{cases}$$

21. $y'' + y = \delta(t), \quad y(0) = 1, \quad y'_-(0) = -2$

22. $y'' - 4y = 3\delta(t), \quad y(0) = -1, \quad y'_-(0) = 7$

23. $y'' + 3y' + 2y = -5\delta(t), \quad y(0) = 0, \quad y'_-(0) = 0$

24. $y'' + 4y' + 4y = -\delta(t), \quad y(0) = 1, \quad y'_-(0) = 5$

25. $4y'' + 4y' + y = 3\delta(t), \quad y(0) = 1, \quad y'_-(0) = -6$

In Exercises 26-28, solve the initial value problem

$$ay_h'' + by_h' + cy_h = \begin{cases} 0, & 0 \le t < t_0, \\ 1/h, & t_0 \le t < t_0 + h, \\ 0, & t \ge t_0 + h, \end{cases} \qquad y_h(0) = 0, \quad y_h'(0) = 0,$$

where $t_0 > 0$ *and* $h > 0$*. Then find*

$$w = L^{-1}\left(\frac{1}{as^2 + bs + c}\right)$$

and verify Theorem 8.7.1 by graphing w *and* y_h *on the same axes, for small positive values of* h*.*

26. L $y'' + 2y' + 2y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

27. L $y'' + 2y' + y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

28. L $y'' + 3y' + 2y = f_h(t), \quad y(0) = 0, \quad y'(0) = 0$

29. Recall from Section 6.2 that the displacement of an object of mass m in a spring–mass system in free damped oscillation is

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

and that y can be written as

$$y = Re^{-ct/2m}\cos(\omega_1 t - \phi)$$

if the motion is underdamped. Suppose $y(\tau) = 0$. Find the impulse that would have to be applied to the object at $t = \tau$ to put it in equilibrium.

30. Solve the initial value problem. Find a formula that does not involve step functions and represents y on each subinterval of $[0, \infty)$ on which the forcing function is zero.

(a) $y'' - y = \sum_{k=1}^{\infty} \delta(t - k), \quad y(0) = 0, \quad y'(0) = 1$

(b) $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \quad y(0) = 0, \quad y'(0) = 1$

(c) $y'' - 3y' + 2y = \sum_{k=1}^{\infty} \delta(t - k), \quad y(0) = 0, \quad y'(0) = 1$

(d) $y'' + y = \sum_{k=1}^{\infty} \delta(t - k\pi), \quad y(0) = 0, \quad y'(0) = 0$

8.8 A BRIEF TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s > 0)$
t^n $(n = \text{ integer } > 0)$	$\frac{n!}{s^{n+1}}$	$(s > 0)$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{(p+1)}}$	$(s > 0)$
e^{at}	$\frac{1}{s-a}$	$(s > a)$
$t^n e^{at}$ $(n = \text{ integer } > 0)$	$\frac{n!}{(s-a)^{n+1}}$	$(s > 0)$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$(s > 0)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$(s > 0)$
$e^{\lambda t} \cos \omega t$	$\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$	$(s > \lambda)$
$e^{\lambda t} \sin \omega t$	$\frac{\omega}{(s-\lambda)^2+\omega^2}$	$(s > \lambda)$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$(s > \vert b \vert)$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$(s > \vert b \vert)$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$(s > 0)$

$t \sin \omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$(s > 0)$
$\sin \omega t - \omega t \cos \omega t$	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$	$(s > 0)$
$\omega t - \sin \omega t$	$\dfrac{\omega^3}{s^2(s^2+\omega^2)^2}$	$(s > 0)$
$\dfrac{1}{t} \sin \omega t$	$\arctan\left(\dfrac{\omega}{s}\right)$	$(s > 0)$
$e^{at} f(t)$	$F(s-a)$	
$t^k f(t)$	$(-1)^k F^{(k)}(s)$	
$f(\omega t)$	$\dfrac{1}{\omega} F\left(\dfrac{s}{\omega}\right), \quad \omega > 0$	
$u(t-\tau)$	$\dfrac{e^{-\tau s}}{s}$	$(s > 0)$
$u(t-\tau) f(t-\tau)\ (\tau > 0)$	$e^{-\tau s} F(s)$	
$\int_o^t f(\tau) g(t-\tau)\, d\tau$	$F(s) \cdot G(s)$	
$\delta(t-a)$	e^{-as}	$(s > 0)$